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ON THE STABILITY OF ACCRETION DISKS

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ABSTRACT. In this paper the stability of standard α accretion disks against perturbations in the radial direction is taken in the light of linear perturbation theory.

This analysis starts from slightly different, but correct, continuity and momentum equations, besides imposing null boundary conditions on the perturbations and continuity along the flow. With such a treatment, we do obtain more stable disks, leading to some modifications in the results earlier obtained by Shakura and Sunyaev (1975).

I. Dynamics of the accretion disks.

The flow in the disk is described by cylindrical polar coordinates, such that z = 0 is the plane of the disk.

As usual, we make the following approximations:

a - the ϕ component of the velocity is Keplerian

b - the disk is thin, i.e., $l \ll r$, $l \equiv$ semi-scale height of the disk
$r \equiv$ radial distance

c - hydrostatic equilibrium in the z-direction

d - the energy radiative transport is only in the z-direction.

If we define U(r,t), surface density, as

$$U = 2\int_0^{\ell} \rho dz$$

where ρ is the volumetric density and $W_{r\phi}$, the viscous stress as

$$W_{r\phi} = 2\int_0^{\ell} \omega_{r\phi} dz \qquad (\omega_{r\phi} \equiv \text{volumetric viscous stress,})$$

we may write the continuity equation,

$$\frac{\partial}{\partial t} U + \frac{\ell}{r}\frac{\partial}{\partial r}\left(\frac{UrV_r}{\ell}\right) = 0$$

or

$$\frac{\partial U}{\partial t} - \frac{\ell}{2\pi r} \frac{\partial}{\partial r} \left(\frac{\dot{M}}{\ell} \right) = 0$$ (I-1), where V_r and $\dot{M} = - 2\pi U_r V_r \equiv$ are respectively the radial velocity and the accretion rate; and the momentum equation as

$$\frac{\dot{M}\Omega r}{2} = 2\pi \ell \frac{\partial}{\partial} \left(\frac{W_{r\phi} r^2}{\ell} \right) \qquad \text{(I-2)}$$

where Ω is the Keplerian angular velocity.

From the hydrostatic equilibrium we have $p = \frac{U\Omega^2 \ell}{6}$ (I-3), where p is the total pressure, and from the very definition of α - model $W_{r\phi} = 2\alpha p \ell$ (I-4), which implies $Q^+ = \frac{3}{2} \alpha p \ell \Omega$ (I-5), where α is the parameter of the model.

Proceeding in this way we get the same energy equation as in the paper by Shakura and Sunyaev

$$(\varepsilon + p) \frac{\partial \ell}{\partial t} + \ell \frac{\partial \varepsilon}{\partial t} = - \frac{\ell}{r} \frac{\partial}{\partial r} \left\{ (\varepsilon + p)\, r \ell V_r \right\} + V_r \frac{\partial}{\partial r} (p\ell) + Q^+ - Q^- \qquad \text{(I-6)}$$

where Q^+, Q^- and ε are respectively the heat production, the energy removal by radiation and the total internal energy, i.e., includes contributions from matter and radiation, as well .

Using equations (1), (2), (3) and (5) we easily rewrite the energy equation as

$$\frac{5 + 3\beta}{12} U\Omega^2 \ell \frac{\partial \ell}{\partial t} + \frac{\ell}{4} \frac{\partial}{\partial t} (1 + \beta) U\Omega^2 \ell = \frac{2\alpha}{3r} \frac{\partial}{\partial r} \left\{ \left(\frac{5 + 3\beta}{12} \right) \frac{\Omega \ell^3}{r} \frac{\partial}{\partial r} (U \ell \Omega^2 r^2) \right\} +$$

$$+ V_r \frac{\partial}{\partial r} (p1) + \frac{\alpha U \Omega^3 \ell^2}{4} - Q^- \qquad \text{(I-7)}$$

where have used

$$\varepsilon = \frac{3}{2} (1 + \beta)\, p$$

$$= \frac{1}{4} (1 + \beta)\, U\Omega^2 \ell \qquad \text{(I-8)}$$

II. Linearization procedure

As Shakura and Sunyaev when linearizing the continuity and energy equation we only retain terms of the order of $(\ell_o/\lambda)^2$, where λ is the wavelength of the perturbation.

If we define the perturbal variables in terms of unperturbed ones

$$U = U_o\ (1 + u)$$
$$\ell = \ell_o\ (1 + h) \qquad \text{(II-1)}$$

where $\underline{0}$ stands for unperturbed variables, the equation of continuity reads

$$\frac{\partial}{\partial t} u = \frac{2}{3} \alpha \Omega \frac{\partial^2}{\partial r^2} (u+h) \qquad \text{(II-2)}$$

Adopting anologous procedure for the energy equation, we get

$$\frac{5 + 3\beta_o}{12} U_o \Omega^2 \ell_o^2 \frac{\partial}{\partial t} h + \frac{U_o \Omega^2 \ell_o^2}{4} \frac{\partial}{\partial t} \left\{ (1+\beta_o)\ (u+h) + \frac{\partial}{\partial t} \beta_1 \right\} =$$

$$= \frac{5 + 3\beta_o}{18} \alpha U_o \Omega^3 \ell_o^4 \frac{\partial^2}{\partial r^2} (u+h) + \frac{\alpha U_o \ell_o^2}{4} \left\{ u+2h- \frac{SQ^-}{Q^-} \right\} \qquad \text{(II-3)}$$

where β_1 is the variation of the ratio of radiation pressure to total pressure . To obtain (II-3) we have used the equality of heat production and energy removal in the unperturbed flow.

III. The energy equation for specific spectral regime

Let us now assume that the disk is a black body in the outer region and optically thin in the inner region, close to R_1. With these assumptions we are able to assign expressions for β_1 and δQ^- .

In the outer region

$$K_{ff} << \tau_T ,$$

$$p = p_g + p_r$$

$$= \frac{UK}{m_H \ell} T + \frac{\pi}{2C} K_R U \tau_B T^4 \qquad \text{(III-1)}$$

where the first term stands for the gas pressure (electrons plus protons) and the second for the radiation :

K_{ff} ≡ free-free opacity

K_R ≡ Rosseland opacity mean (cm^{-2} g^{-1})

$$K_R = \frac{AU}{\ell} T^{-7/2} \tag{III-2}$$

A ≡ cte. , T ≡ temperature

τ_B ≡ Stefan - Boltzmann constant

$$Q^- = \frac{c}{\pi\tau} p_r = \frac{c}{\pi\tau} \beta p \tag{III-3}$$

where τ is the effective optical depth.

From these equations we easily get

$$\frac{\beta_1}{\beta_o} = + 2 \left(\frac{1-\beta_o}{3-2\beta_o}\right) h + 2 \left(\frac{1-\beta_o}{3-2\beta_o}\right) u \tag{III-4}$$

$$\frac{\delta T}{T_o} = 4h \left(\frac{2-\beta_o}{3-2\beta_o}\right) - \frac{2u}{3-2\beta_o} \tag{III-5}$$

where $\frac{\delta T}{T_o}$ is ratio of the variation of the temperature to the unperturbed temperature.

Inserting III-3,3 and 5 into the energy equation (II-3), we get, after some algebra

$$\left\{(24 - 13\beta_o - 6\beta_o^2)\ \omega^2 + 3\alpha\Omega\ (26 - 12\beta_o)\ \omega\right\} y =$$

$$= \frac{2}{3} \alpha\Omega\ell_o^2 \left\{ (30 - 14\beta_o)\ \omega + 3\alpha\Omega\ (37 - 15\beta_o) \right\} y \tag{III-6}$$

where we have put $y = u+h$, and assumed $y \alpha \exp \omega t$.

The solution of (III-6) in the small wavelength limit is $y \alpha$ sen $\frac{2\pi r}{\Lambda}$ which gives the dispersion relation.

$$(24 - 13\beta_o - 6\beta_o^2)\,\omega^2 + 3\alpha\Omega\omega\,[(26 - 12\beta_o) + \frac{2}{3}(l_o K)^2(30 - 14\beta_o)] + 2\,(\alpha\Omega l_o K)^2(37 - 15\beta_o) = 0 \qquad (III-7)$$

With null boundary condition in the outer radius R_2, K is given by

$$KR_2 = \pi\,(2n + 1)$$

$$K = \frac{\pi(2n + 1)}{R_2} \qquad (III-8)\ ,\ n = \text{integer}$$

From (III-7) we have an eigenvalue equation for K.

$$(\ell_2 K)^2 = -\frac{(24 - 13\beta_o - 6\beta_o^2)\,\omega^2 + 3\alpha\Omega\omega\,(26 - 12\beta_o)}{2\alpha\Omega\omega\,(30 - 14\beta_o) + 2(\alpha\Omega)^2\,(37 - 15\beta_o)} \qquad (III-9)$$

where ℓ_2 stands for the scale height at $r = R_2$

In inner region

$$Q^- = \frac{c\beta p}{\pi\tau_T U} \qquad (III-10)$$

$$\nu_r = \frac{AU^3 T^{1/2}}{\ell} \qquad (III-11)$$

This gives

$$\frac{\beta_1}{\beta_o} = 2\left(\frac{1 - \beta_o}{2 - \beta_o}\right)(2u - h) \qquad (III-12)$$

$$\frac{\delta T}{T_o} = \frac{4h}{2-\beta_o} - \frac{4\beta_o}{2-\beta_o}\,u \qquad (III-13)$$

Inserting these relation into (II-3) we get

$$\left\{(16 - 2\beta_o)\,\omega^2 - 3\alpha\Omega\omega\,(4 - 3\beta_o)\right\} y = \frac{2}{3}\,\alpha\Omega\ell_o^2\,\frac{\partial^2}{\partial r^2}\,y\,\left\{(20 + 20\beta_o - 24\beta_o^2)\,\omega - 18\,\alpha\Omega\,(1 - \beta_o)\right\} \qquad (III-14)$$

Putting $y\alpha$ sen K_r yields the dispersion relation

$$(16 - 2\beta_o)\,\omega^2 - \alpha\Omega\omega\,[3(4 - 3\beta_o) = \frac{2}{3}\,(\ell_o K)^2(20 + 20\beta_o - 24\beta_o^2)] - 12\,(\alpha\Omega l_o K)^2(1-\beta_o) = 0 \qquad (III-15)$$

and the eigenvalue equation for K,

$$(\ell_1 K)^2 = \frac{(16 - 2\beta_0)\,\omega^2 - 3\alpha\Omega\omega\,(4 - 3\beta_0)}{12(\alpha\Omega)^2(1-\beta_0) - \frac{2}{3}\,\alpha\Omega\omega\,(20+20\beta_0-24\beta_0^2)} \qquad \text{(III-16)}$$

$$\ell_1 = \ell(r=R_1)$$

Equating (III-9) and (III-16) and assuming $\beta_0 \cong 0$ at R_2 and $\beta_0 \cong 1$ at R_1, we get

$$\lambda^3 \left\{ \frac{f^2 840}{x^{3/2}} - 256 \right\} + \lambda^2 \left\{ f^2 \frac{1036}{x^3} - \frac{180}{x^{3/2}} - \frac{896}{x^{3/2}} \right\}$$

$$- \frac{222 f^2}{x^3}\,\lambda = 0 \qquad \text{(III-17)}$$

where $x = \frac{R_2}{R_1}$ is the disk length and $f = \frac{\ell_2}{\ell_1}$

and $\lambda = \frac{\omega}{\alpha\Omega(R_1)}$

Using now the hydrostactic equilibrium equation (assuming $\beta_0 = 1$ at R_1 and $\beta_0 = 0$ at R_2) we have in the outer region

$$\ell_2 = 1.78 \times 10^{-23}\,\dot{M}^{1/8}\,M^{3/4}\,x^{9/8} \qquad \text{(III-18)}$$

$$\ell_1 = 2.87 \times 10^{-11}\,\dot{M}\,(1 - \delta) \qquad \text{(III-19)}$$

where δ is the ratio of the angular momentum of the flow the Keplerian angular momentum at $r = R_1$. Assuming $\delta = 0.9$, we have

$$\ell_1 = 2.87 \times 10^{-12}\,\dot{M}, \quad \text{and}$$

$$f = 6.2 \times 10^{-12}\,\frac{M^{3/4}}{\dot{M}^{7/8}}\,x^{9/8} \qquad \text{(III-20)}$$

here M, $\dot{M}$ are respectively the mass of the central compact object and the accretion rate.

Equation (III-17) becomes, omitting the $\lambda = 0$ solution,

$$\lambda^2\{3.22 \times 10^{-18} q\, x^{3/4} - 256\} + \lambda\{3.85 \times 10^{-21}\, q\,[\frac{1036}{x^{3/4}} \times 180\, x^{3/4}]\}$$

$$- \frac{896}{x^{3/2}} = 0 \qquad \text{(III-21)}$$

$$q \equiv \frac{M^{3/2}}{\dot{M}^{7/4}}$$

IV. Results and Conclusions

An inspection of equation (III-21) reveals that perturbations will have their growth damped as long as the mass, the accretion rate and the length of the disk satisfy the relation

$$\frac{M^{3/2}}{\dot{M}^{7/4}} x^{3/4} \leq 8 \times 10^{18} \qquad \text{(IV-1)}$$

Obviously besides this, for stability it is necessary that the disk emits like a black body at the outer region. Equating scattering to free-free opacity at the transition region x_t, we have

$$T^{7/2} = 3.5 \times 10^{25} \rho \qquad \text{(IV-2)}$$

From $Q_o^- = Q_o^+$

$$T^{7/2} = 4.84 \times 10^{68} M^{-7/4} \dot{M}^{7/8} x_t^{-21/8} \qquad \text{(IV-3)}$$

Equating (IV-2) and (IV-3) we get

$$x_t = 1.3 \times 10^{12} \alpha^{8/33} M^{-2/3} \dot{M}^{23/33} \qquad \text{(IV-4)}$$

if radiation pressure still dominates at x_t, or

$$x_t = 8.1 \times 10^{17} M^{-2/3} \dot{M}^{1/3} \alpha^{4/3} \qquad \text{(IV-5)}$$

if gas pressure dominates at x_t.

In the inner region $\tau_{eff} < 1$, which means

$$\alpha < 2.29 \times 10^{16} \frac{M^{1/4}}{\dot{M}^{4/3}} \qquad \text{(IV-6)}$$

If we now apply these relations for a disk with a critical accretion rate, we get

$$x < 1.6 \times 10^{-1/3} M^{1/3} \qquad \text{(IV-7)}$$

which is the largest disk length, not compatible with the usually assumed x=100

For α we get

$$2.15 \times 10^{36}\ M^{-13/12} \leq \alpha \leq \begin{cases} 9.45 \times 10^{-18}\ M^{1/2} \\ 5.6 \times 10^{-48}\ M^{10/8} \end{cases} \qquad \text{(IV-8)}$$

where the first refers to $p_r >> p_g$ at x_t and the second to $p_r << p_g$ at x_t.

It is worth to remark that these conditions do allow for supercritical disks.

REFERENCES

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