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PERIOD ANALYSIS OF LAPLACEAN RESONANCE AMONGST URANIAN INNER SATELLITES

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ABSTRACT. The study of the inequalities in the mean longitudes of Uranian satellites is used to determine optimized initial conditions for a numerical integration of resonance equations.

I. INTRODUCTION.

The study of the Laplacean resonance amongst Uranian inner satellites permits the determination of the orbital parameters of these satellites. Recent works (Lazzaro et al. 1984) based on the analysis of this resonance suggest that the masses of the satellites are directly responsible for the period of resonance. The present work intends to extend the analysis of the resonance mainly through the numerical integration of the inequalities for the satellites mean longitudes. An exaustive analysis of the solution is given together with a study of the main influence on the resonance period.

II. LAPLACIAN RESONANCE

The inequalities in Uranus' satellites mean longitude, due to the near - commensurability amongst Miranda, Ariel and Umbriel, can be written for each satellite as

$$\frac{d^2 \lambda_i}{dt^2} = n_i^2 \, m_j \, m_k \, (C_i' \sin \Theta + C_i'' \sin 2\Theta) \qquad j \neq K \neq i \qquad (1)$$

where n_i is the observed mean motion of the i-th satellite, m_i its mass, C_i' and C_i'' its coefficients of resonance and Θ is the critical relation among the mean longitudes given by

$$\Theta = \lambda_1 - 3\lambda_2 + 2\lambda_3$$

where the subscripts number the satellites in order of increasing distance from the planet. The coefficients $C_i^{'}$ and $C_i^{''}$, given in units of mass of Uranus, are functions of the semi-major axis and the mean motion of the satellites. These coefficients have been computed by means of the equations of the classical Laplace-Suillart theory of the motion of the Galilean satellites of Jupiter (Tisserand, 1986; Ferraz Mello, 1979) adapted to the particularities of Uranus' satellites system (Lazzaro et al, 1984).

In order to analyze relation (1) we performed the numerical integration of the differential system

$$\frac{dn_i}{dt} = A_i \sin \Theta + B_i \sin 2\Theta \qquad (2)$$

$$\frac{d\lambda_i}{dt} = n_i$$

with i=1, 2 and 3. The results were analyzed by means of a discrete Fourier transform as given by Ferraz Mello (1981) adapted to uniform spacing.

The numerical integration of system (2) was initially performed with the following conditions at t_o:

$$\lambda_{io} = \lambda_i(t_o)$$

$$n_{io} = \frac{2\pi}{P_i} \qquad (3)$$

The values used for $\lambda_i(t_o)$, P_i and the satellites' masses are given in table I.

TABLE I

Satellite elements J. D. 2432800.0

Satellites	Mass (M_u)	Period (d)	Longitude	$C'(M_u^{-2})$	$C''(M_u^{-2})$
Miranda	0.2×10^{-5}	1.41347925	1.9627972	-11.50	+ 5.83
Ariel	1.8×10^{-5}	2.52037935	4.2350936	+50.79	-25.87
Umbriel	1.2×10^{-5}	4.1441772	0.6276383	-47.15	+24.16

The results present however a period of resonance of 4370 days which. is quite different from the period we obtain using

$$P_r = \frac{P_1 P_2 P_3}{P_2 P_3 - 3 P_1 P_3 + 2 P_1 P_2} \qquad (4)$$

that is 4587 days.

This occurs because we have modified the mean motion and consequently the revolution period of each satellite.

Letus consider

$$\lambda_{io} = \lambda_i(t_o)$$
$$n_{io} = \frac{2\pi}{P_i} - \Delta n_i \qquad (5)$$

where again $\lambda_i(t_o)$ and P_i are from table I and Δn_i are arbitrary coefficients. These coefficients can be obtained by the following iterative process: in a first order aproximation we set Δn_i equal to zero and determine a linear coefficient B_1 which is introduced as Δn_i in a new integration. In this way a smaller secular part is obtained and consequently a smaller B_2; this one added to B_1 can be used in a new integration giving linear coefficients that converge to zero. This iterative process was used until linear coefficients less than 10^{-8} degrees were obtained, usually in the 3^{th} or 4^{th} integration.

Relation

$$\lambda_i(t) = \lambda_i(t_o) + \xi_i + n_i(t-t_o) + A_i' \sin\left[\psi_o - \frac{2\pi}{P_r}(t-t_o)\right] +$$
$$- A_i'' \sin 2\left[\psi_o - \frac{2\pi}{P_r}(t-t_o)\right] \qquad (6)$$

where

$$\zeta_i = -A_i' \sin \psi_o + A_i'' \sin 2\psi_o$$

describes the new results obtained and the initial conditions are satisfied only if

$$\Delta n_i = \frac{2\pi}{P_r}\left[A_i' \cos \psi_o - 2 A_i'' \cos 2\psi_o\right] \qquad (7)$$

This relation can be verified using the results given in table II where we can also see that the period of resonance is now equal to the espected value, considering the precisions involved. The phase ψ_0 in relation (7) is equal to the critical relation between the mean longitudes at t_0 as can be deduced equating relation (1) with the derivatives of the relation for λ_i (t).

TABLE II

Results of the numerical integration and spectral analysis with initial conditions (10)

			Relation (6)		
Sat.	Masses (10^{-5} M_u)	Δn (10^{-3})	A'	A''	P_r(d)
1 2 3	0.4 1.8 1.2	$2^{\circ}.559$ $-\ 0^{\circ}.79083$ $0^{\circ}.40779$	$1^{\circ}.5387$ $0^{\circ}.4750$ $0^{\circ}.2447$	$0^{\circ}.1765$ $0^{\circ}.0547$ $0^{\circ}.0284$	4590
1 2 3	0.2 1.8 1.2	$2^{\circ}.56145$ $-\ 0^{\circ}.395729$ $0^{\circ}.204085$	$1^{\circ}.5285$ $0^{\circ}.2359$ $0^{\circ}.1215$	$0^{\circ}.1801$ $0^{\circ}.0279$ $0^{\circ}.0145$	4590
1 2 3	0.05 1.8 1.2	$2^{\circ}.5627$ $-\ 0^{\circ}.099$ $0^{\circ}.051045$	$1^{\circ}.5208$ $0^{\circ}.0586$ $0^{\circ}.0302$	$0^{\circ}.1827$ $0^{\circ}.0071$ $0^{\circ}.0037$	4590
1 2 3	0.2 2.2 0.8	$2^{\circ}.0878$ $-\ 0^{\circ}.26394$ $0^{\circ}.24951$	$1^{\circ}.2413$ $0^{\circ}.1567$ $0^{\circ}.1480$	$0^{\circ}.1481$ $0^{\circ}.0188$ $0^{\circ}.0179$	4590
1 2 3	0.2 1.4 1.6	$2^{\circ}.6557$ $-\ 0^{\circ}.5276$ $0^{\circ}.15865$	$1^{\circ}.5878$ $0^{\circ}.3151$ $0^{\circ}.0947$	$0^{\circ}.1859$ $0^{\circ}.0370$ $0^{\circ}.0112$	4590

Due to fact that the masses of Uranus' satellites are poorly determined, mainly Miranda's, we applied the method just described to several sets of masses and obtained the remarkable result that the period of resonance does not depend on small variations in the satellites' masses.

III. CONCLUSIONS

The results obtained by the numerical integration of the differential equations for Laplacean resonance clearly show that the initial mean motions have more influence on the period of resonance than satellites' masses, A numerical integration of the equations of motion of the complete Uranus' satellite system is being carried out at the Observatório Nacional in order to check the results here presented.

REFERENCES

Ferraz Mello, S. (1979): Dynamics of the Galilean Satellites: an introductory treatise, Univ. São Paulo, São Paulo.

Ferraz Mello, S. (1981): Astron. J. 86, 619.

Lazzaro, D. (1982): M. Sc. Thesis, Univ. São Paulo, São Paulo.

Lazzaro, D., Ferraz Mello, S. and Veillet, C. (1984): Astr. Astrophys. 140, 33.

Tisserand, F. (1896): Traité de Mecanique Celeste, Gauthier Villars, Paris, V 4.

Veillet, C. (1983a): Astron. Astrophys. 118, 211.

Veillet, C. (1983b): Doctor thesis, Université de Paris 6.

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