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THERMAL RELAXATION TIME OF A MIXTURE OF RELATIVISTIC ELECTRONS AND NEUTRINOS

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RESUMEN. La interacción entre las componentes de una mezcla binaria de gases relativistas se estudia por medio de un formalismo totalmente covariante. Partiendo de la ecuación de Boltzmann relativista se derivan expresiones generales para la tasa de intercambio de energía entre las componentes, supuestas a temperaturas ligeramente diferentes, y para el tiempo de relajamiento del sistema. La fórmula resultante se aplica a una mezcla de electrones relativistas y neutrinos y se obtienen resultados numéricos para su tiempo de relajamiento.

ABSTRACT. The interaction between the components of a relativistic binary mixture is studied by means of a fully covariant formalism. Assuming both components to differ slightly in temperature, an application of the relativistic Boltzmann equation yields general expressions for the energy transfer rate and for the relaxation time of the system. The resulting relation is then applied to a mixture of relativistic electrons and neutrinos to obtain numerical values of its relaxation time.

Key words: RELATIVITY

INTRODUCTION

The theoretical study of non-equilibrium relativistic systems is acquiring importance in our days due to the almost certain presence of relativistic gases in several astronomical phenomena, such as jets, active nuclei, accretion disks around black holes and, obviously, the early universe. This kind of work became possible only after a wholly covariant relativistic kinetic theory was accomplished (full accounts of the theory are given by Stewart 1971; de Groot 1974; Ehlers 1974 and de Groot et al. 1980). The formalism has been applied to derive a general expression for the relaxation time toward thermal equilibrium of a binary mixture of ideal relativistic gases at slightly different temperatures (Herrera and Hacyan 1985, hereafter referred to as paper I). In the present work, the general expression is applied to the particular case of a mixture of relativistic electrons and neutrinos, assumed to interact only through elastic collisions. The Weinberg-Salam theory is fully taken into account and numerical values of the thermal relaxation time of the mixture are obtained for electron temperatures in the most relevant range, i.e., $6 \times 10^8\ K \leq T \leq 10^{11} K$.

THEORY

For the sake of completeness, a brief derivation of the expression for the relaxation time is given in this section. The reader interested in the more detailed approach is referred to paper I.

Consider, then, a system composed by two kinds of particles, 1 and 2 say. Each kind of particle is described by a Lorentz invariant distribution function f_i (i = 1,2) which satisfies the relativistic Boltzmann equation

$$p_i{}^{\mu} f_{i,\mu} = \sum_j C_{ij} \qquad (1)$$

where p_i^{μ} is the four-momentum of particles i and the C_{ij} are the so-called "collision terms", given by Eq. (2) of paper I. In thermodynamic equilibrium

$$f_i = \left\{ \exp\left[\beta (p_i \cdot U) - \beta\mu_i\right] - \varepsilon_i \right\}^{-1} \tag{2}$$

where $\beta \equiv (k_B T)^{-1}$, k_B being Boltzmann's constant, U^{μ} is the hydrodynamic four-velocity, μ is the relativistic chemical potential of particles i and $\varepsilon_i \doteq -1,0,1$ according to whether fermions, classical particles or bosons, are considered. In terms of the f_i, the energy-momentum tensor of particles i is given by

$$T_i^{\mu\nu} = \frac{c\, g_i}{h^3} \int p_i^{\mu} p_i^{\nu} f_i \, d\Gamma_i \tag{3}$$

where g_i is the occupation number in phase space of particles i, h is Planck's constant and $d\Gamma_i$ is the invariant momentum-space element given by equation (3) of paper I. From the conservation of the total energy-momentum of the system ($T^{\mu\nu}{}_{,\nu} = 0$; $T^{\mu\nu} = T_1^{\mu\nu} + T_2^{\mu\nu}$) it follows that, at constant volume,

$$\dot{e}_1 + \dot{e}_2 = 0 \tag{4}$$

where e_i is the mass-energy density of particles i and the dot represents the derivative along U^{μ} ($\dot{} \equiv U^{\mu}\partial_{\mu}$). Eq. (4) simply states the obvious fact that the total mass-energy of the system is conserved.

Assume now that each component is in thermodynamic equilibrium with itself, at a temperature T_i, and that both temperatures differ only slightly, i.e.,

$$\begin{aligned} T_1 &= T + \Delta T \\ T_2 &= T \end{aligned} \tag{5}$$

with $\Delta T \ll T$. It may be shown (see paper I) that in this case we have, to first order in ΔT,

$$\dot{e}_1 = -\dot{e}_2 = -I\Delta T \tag{6}$$

where

$$I = \frac{g_1 g_2 c}{h^6 k_B T^2} \int d\Gamma_1 d\Gamma_2 d\Gamma_1' d\Gamma_2' \; f_1' f_2' \hat{f}_1 \hat{f}_2 (p_1 \cdot U)\left[(p_1 - p_1') \cdot U\right](p_1 + p_2)^2 \sigma$$

$$\delta^{(4)}(p_1 + p_2 - p_1' - p_2') \tag{7}$$

In this equation $\hat{f}_i \equiv 1 + \varepsilon_i f_i$, $\delta^{(4)}$ is Dirac's delta function in 4 dimensions and σ is the cross section, in the center of momentum frame, of the microscopic interaction between both kinds of particles responsible for their approach to equilibrium.

On the other hand, we also have the obvious relation

$$\dot{e}_i = C_i \dot{T}_i \tag{8}$$

where

$$C_i = \left(\frac{\partial e_i}{\partial T}\right)_n \tag{9}$$

is the heat capacity per unit volume, at constant volume, of particles i, n being the (conserved) baryon number density. From eqs. (4), (5), (6) and (8) we finally obtain for the thermal relaxation time τ:

$$\tau \equiv \frac{\Delta T}{\dot{\Delta T}} = \frac{C_T}{I} \tag{10}$$

where we have defined

$$C_T^{-1} = C_1^{-1} + C_2^{-1} \tag{11}$$

ELECTRON-NEUTRINO MIXTURE

We now apply eq. (10) to the particular case of components 1 and 2 being electrons and neutrinos, respectively. Let $p_1 = p$ and $p_2 = q$. Then

$$\begin{aligned} f_1 \equiv f &= \left\{\exp(\beta p \cdot U) + 1\right\}^{-1} \\ f_2' \equiv N &= \left\{\exp(\beta q \cdot U) + 1\right\}^{-1} \end{aligned} \tag{12}$$

where both chemical potentials have been taken equal to zero for simplicity. The cross section for elastic electron-neutrino scattering is (de Groot et al. 1980, p. 319)

$$\sigma = \frac{\pi^2 G^2}{c^2 h^4} \frac{(P^2 - m^2c^2)^2}{p^2} \left\{ C(1+C)(1+\cos\theta) + \left[1 + \frac{C}{2} \frac{P^2 - m^2c^2}{p^2} (1 - \cos\theta)\right]^2 \right\} \tag{13}$$

where θ is the scattering angle in the center of momentum frame, $P = p + q$ is the total (conserved) four-momentum, $G = 1.435 \times 10^{-49}$ gr·cm^5/seg^2 is the weak interaction constant and $C = 4 \sin^2\theta_W$, with θ_W = Weinberg angle. From now on we will take $C = 1$ (de Groot et al. 1980, p. 331).

When all these relations, together with the known values $g_1 = 2$, $g_2 = 1$, are substituted in eq. (10), an expression for I is obtained, as a 16-fold integral over all the four-momenta. Nevertheless, 12 of these integrations may be performed analytically applying a procedure fully described in paper I and in chapter XIII of de Groot et al. (1980). The main idea is to integrate over the scattering angles in the center of momentum frame and to integrate over four-momenta magnitudes in the rest frame. Since the algebra is quite long and cumbersome, we will simply write the final expression, i.e., the 4-fold integral which must be evaluated numerically, and which turns out to be:

$$I = \frac{\pi^3 c k_B \sigma_0}{4} \left(\frac{mc}{h}\right)^6 \phi^2 \int_1^\infty dv \frac{(v^2-1)^6}{v^3} \int_0^\infty d\omega \sinh^4\omega H(a,b,c) \tag{14}$$

where $\sigma_0 \equiv \left(\frac{\pi G m}{2h^2}\right)^2$, $\phi \equiv \frac{mc^2}{k_B T}$, m being the electron mass,

$$H(a,b,c) = \int_{-1}^{1} \frac{dx}{\cosh a + \cosh(b - cx)} \int_{-1}^{1} \frac{dx'}{\cosh a + \cosh(b - cx')} \times (x - x') \tag{15}$$

$$\left[A(3x^2x'^2 - x^2 - x'^2 + 1) + 2Bxx' + 2D\right]$$

with $A = \alpha^2$, $B = 2(\gamma - 2\alpha - \alpha^2)$, $D = 2\gamma + (2+\alpha)^2$, $\alpha = \frac{C}{2}\left(\frac{v^2-1}{v^2}\right)$, $\gamma = C\left(1 + \frac{C}{2}\right)$, $a = \frac{1}{2}\phi v \cosh\omega$, $b = a/v^2$, $c = (a-b)\tanh\omega$. For the heat capacity of electrons, we have from Herrera (1986), eq. (19) of that paper,

$$C_e = 4\pi \left(\frac{mc}{h}\right)^3 k_B \phi^2 \int_0^1 \frac{y^2 dy}{(1-y^2)^{7/2}} \left[\cosh\left(\frac{\phi}{\sqrt{1-y^2}}\right) + 1\right]^{-1} . \tag{16}$$

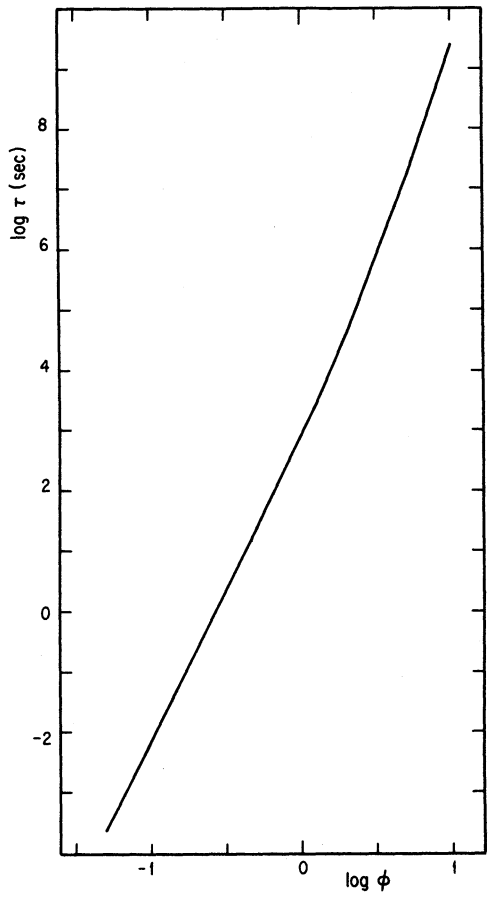

Fig. 1. Thermal relaxation time (in seconds) as a function of the temperature parameter ϕ.

For neutrinos, the heat capacity follows from eq. (3.108) in Chiu (1968), with $\mu = 0$, and it may be written in closed form as

$$C_\nu = 84\pi\left(\frac{mc}{h}\right)^3 k_B \zeta(4)\phi^{-3} \qquad (17)$$

where $\zeta(4) = 1.082323$ is Riemann's ζ-function of argument 4 (see, for instance, Abramowitz, 1968).

Equations (10) and (14)-(17) allow us to evaluate the thermal relaxation time of the mixture.

RESULTS AND CONCLUSIONS

All the integrals were numerically evaluated, with a prescribed precision of 1%, in the temperature range $.05 \leq \phi \leq 10$, which is the range in which future applications to the early universe are intended. The results are plotted in Fig. 1. It is interesting to note that, were there only electrons and neutrinos in the early universe, the relaxation time and the expansion timescale for an Einstein-de Sitter universe ($\tau_{exp} = 4.38\ \phi^2$) would become equal at a "cosmic time" $t = \frac{3}{2}\tau_{exp} \sim 0.2$ sec., a value of the correct order of magnitude since neutrinos are usually taken to decouple at $t \lesssim 1$ sec. This result is only preliminary; in order to obtain a more realistic value, all reactions involving electrons, positrons, neutrinos and antineutrinos should be taken into account (Hacyan and Herrera, in preparation).

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DISCUSSION

IBAÑEZ: Deseo hacer énfasis en la importancia de este tipo de trabajo ya que en publicaciones recientes están apareciendo peligrosas extrapolaciones de resultados no relativistas a plasmas relativistas.

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