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GALACTIC KINEMATICS: AN AXI-SYMMETRIC TIME-DEPENDING MODEL WITH SEPARABLE POTENTIAL

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RESUMEN. Se ha deducido un modelo cinemático galáctico no estacionario, con simetría cilíndrica y potencial separable a partir de las hipótesis de Chandrasekhar, habiéndose obtenido el movimiento de las estrellas a partir de las integrales primeras del sistema. El modelo ha sido aplicado a una muestra de estrellas de clase de luminosidad V y tipos espectrales F6 - F7.

ABSTRACT. An axi-symmetric time-depending kinematic galactic model with separable potential has been derived under Chandrasekhar hypothesis, being the motions of the stars obtained through the first integrals of the system. The model has been applied to a sample of luminosity class V, F6 - F7 spectral type stars.

Key words: GALAXY-KINEMATICS

I. INTRODUCTION

Based in Chandrasekhar (1960) hypotheses for the study of stellar systems, a time-depending axi-symmetric kinematic galactic model has been developed when the galactic potential is assumed to be separable.

Under these hypotheses, an stellar system it is described by a distribution function $f(t,\underline{r},\underline{V})$ that gives the number of stars which in a time t have positions between $\underline{r}$ and $\underline{r} + d\underline{r}$ and velocities between $\underline{V}$ and $\underline{V} + d\underline{V}$. This distribution function is supposed to be of the form

$$f(t,\underline{r},\underline{V}) = f(Q+\sigma)$$

where Q is a quadratic form

$$Q = \underline{v}^T \cdot A \cdot \underline{v}$$

being

$$\underline{v} = \underline{V} - \underline{V}_0$$

the residual velocity of a star,

$$\underline{V}_0(t,\underline{r}) = \frac{1}{N(t,\underline{r})} \int_{\underline{V}} \underline{V}\, f(t,\underline{r},\underline{V})\, d\underline{V}$$

the velocity of the local standard of the rest and $A(t,\underline{r})$ a symmetric second order tensor, and $\sigma(t,\underline{r})$ is a scalar function.

If we write

$$\underline{\Delta} = A \cdot \underline{V}_0$$

$$-\chi = \underline{V}_0^T \cdot A \cdot \underline{V}_0 + \sigma = \underline{\Delta} \cdot \underline{V}_0 + \sigma$$

the fundamental equation of stellar dynamics leads to the twenty scalar equation system described by

$$\text{def } A = 0 \tag{1.1}$$

$$\text{def } \underline{\Delta} = \frac{1}{2}\frac{\partial A}{\partial t} \tag{1.2}$$

$$A \cdot \nabla U + \frac{\partial \underline{\Delta}}{\partial t} = -\frac{1}{2}\Delta\chi \tag{1.3}$$

$$\underline{\Delta} \cdot \nabla U = \frac{1}{2}\frac{\partial \chi}{\partial t} \tag{1.4}$$

where def denotes a generalization (Orús 1952) of the strain operator of the ellasticity theory. The model will therefore be characterized after having determined A, $\underline{\Delta}$, U and χ.

II. DERIVATION OF THE MODEL

Written in cylindrical coordinates, solutions of (1.1) and (1.2) are (Català 1972; Orús 1977)

$$\begin{aligned} A_{\varpi\varpi} &= k_1 + k_4 z^2 \\ A_{\varpi\theta} &= 0 \\ A_{\varpi z} &= -k_4 \varpi z \\ A_{\theta\theta} &= k_1 + k_2\varpi^2 + k_4 z^2 \\ A_{\theta z} &= 0 \\ A_{zz} &= k_3 + k_4\varpi^2 \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} \Delta_{\varpi} &= \frac{1}{2}\dot{k}_1\varpi \\ \Delta_{\theta} &= -\beta\varpi \\ \Delta_z &= \frac{1}{2}\dot{k}_3 z \end{aligned} \tag{2.2}$$

where k_2 and k_4 are positive constants and k_1 and k_3 are positive functions of t. The functions of t usually denoted by k_5, k_6 and δ have been taken identically zero.

Equations (1.3) and (1.4) give the constant character of β and they lead to the determination of U and χ. These functions can be written as

$$U = \frac{B_1}{2\tau} + \frac{B_2}{2\zeta} + \frac{H(t)}{4}(\tau + \zeta) \tag{2.3}$$

$$-\frac{1}{2}\chi = -\frac{B_1 k_1}{2\tau} + \frac{B_2 k_3}{2\zeta} + k_4\left(\frac{B_1\zeta}{\tau} + \frac{B_2\tau}{\zeta}\right) + \frac{H(t)}{4}(k_1\tau + k_3\zeta) +$$

$$+ \frac{1}{2}(\ddot{k}_1\tau + \ddot{k}_3\zeta) + \text{constant} \tag{2.4}$$

where

$$\tau = \frac{1}{2}\varpi^2 \qquad k_1 = \phi_1^2$$

$$\zeta = \frac{1}{2}z^2 \qquad k_3 = \phi_2^2$$

B_1 and B_2 are constants, and

$$\frac{H(t)}{4} = -\frac{\ddot{\phi}_1}{\phi_1} + \frac{\delta_1}{\phi_1^4} = -\frac{\ddot{\phi}_2}{\phi_2} + \frac{\delta_3}{\phi_2^4} \tag{2.5}$$

being also δ_1 and δ_3 constants.

A more detailed description of this process and of the properties of the model can be found in Sala (1986) and it will be published elsewhere in the next future.

III. DESCRIPTION OF THE POTENTIAL

If it is adopted $B_2 = 0$ in order to avoid an infinite value of (2.3) and (2.4) in the galactic plane, and it is taken (Oort 1965)

$$H(t) = 4k^2 > 0$$

where k is a constant, the form of (2.3) and (2.4) will be

$$U = \frac{B_1}{\varpi^2} + \frac{k^2}{2}(\varpi^2 + z^2)$$

$$-\frac{1}{2}\chi = \frac{B_1(k_1 + k_4 z^2)}{\varpi^2} + \frac{k^2}{2}(k_1\varpi^2 + k_3 z^2) + \tag{3.1}$$

$$+ \frac{1}{4}(\ddot{k}_1\varpi^2 + \ddot{k}_3 z^2) + \text{constant}$$

where, from (2.5),

$$k_1 = \phi_1^2 = \alpha_1 \sin(2kt + \beta_1) + \gamma_1$$

$$k_3 = \phi_2^2 = \alpha_3 \sin(2kt + \beta_3) + \gamma_3$$

with

$$\delta_1 > 0 \qquad \delta_3 > 0$$

$$\gamma_1 > 0 \qquad \gamma_3 > 0$$

$$\gamma_1^2 = \alpha_1^2 + \frac{\delta_1}{k^2} \qquad \gamma_3^2 = \alpha_3^2 + \frac{\delta_3}{k^2}$$

The term B_1/ϖ^2 of the potential was described by Jeans (1923) as the potential necessary in order that spiral arms of galaxies, considered as paths of stars, should keep their logarithmic spiral shape with time. The other term $(k^2/2)(\varpi^2 + z^2)$ is the potential at a point inside an homogeneous sphere (Ogorodnikov 1965).

IV. MOTIONS OF STARS

When the potential is assumed to be (3.1), the first integrals of the system are

$$I_1 \equiv \Pi^2 + \theta^2 + \frac{2B_1}{\varpi^2} + k^2\omega^2 = \text{constant}$$

$$I_2 \equiv \varpi\theta = \text{constant}$$

$$I_3 \equiv Z^2 + k^2 z^2 = \text{constant}$$

$$I_4 \equiv (z\Pi - \omega Z)^2 + z^2\theta^2 + \frac{2B_1 z^2}{\varpi^2} = \text{constant}$$

and they can be used to solve the differential system associated to the fundamental equation. The projection on the galactic plane of the paths of star motions will be given by

$$\frac{1}{\varpi^2} = M_1 + M \cos L(\theta - \theta_0)$$

where

$$L = 2\,\frac{(I_2^2 + 2B_1)^{\frac{1}{2}}}{I_2^2} \qquad M_1 = \frac{I_1}{2(I_2^2 + 2B_1)} \qquad M^2 = M_1^2 - \frac{4k^2}{I_2^2 L^2}$$

and whose apocenter and pericenter are

$$\frac{1}{\varpi_A^2} = M_1 - M \qquad \frac{1}{\varpi_p^2} = M_1 + M$$

Radial and perpendicular motions of stars are periodic and they are described by

$$\varpi^2 = \frac{M_1}{M_1^2 - M^2} - \frac{M}{M_1^2 - M^2}\cos(2kt + 1)$$

$$z = D \cos(kt + m)$$

with

$$D = \frac{I_3^{\frac{1}{2}}}{k}$$

as well as the path on a meridian plane rolling with each star is

$$I_3^2\varpi^4 - 2\,(I_1 I_3 - 2k^2 I_4)\varpi^2 z^2 + (I_1^2 - 4k^2 I_2^2 - 8k^2 B_1)\, z^4 -$$

$$- 2I_3 I_4 \varpi^2 - 2\,(I_1 I_4 - 2I_2^2 I_3 - 4B_1 I_3)\, z^2 + I_4^2 = 0$$

The motion of the local standard of the rest is given by

$$dt = \frac{d\varpi}{\Pi_0} = \frac{d\theta}{\theta_0/\varpi} = \frac{dz}{Z_0}$$

where, from (2.1) and (2.2),

$$\Pi_0 = \frac{1}{2}\,\frac{\dot{k}_1 k_3 + \dot{k}_1 k_4 \varpi^2 + \dot{k}_3 k_4 z^2}{k_1 k_3 + k_1 k_4 \varpi^2 + k_3 k_4 z^2}$$

$$\theta_0 = -\,\frac{\beta\varpi}{k_1 + k_2\varpi^2 + k_4 z^2} \qquad (4.1)$$

$$Z_0 = \frac{1}{2}\,\frac{\dot{k}_3 k_1 + \dot{k}_1 k_4 \varpi^2 + \dot{k}_3 k_4 z^2}{k_1 k_3 + k_1 k_4 \varpi^2 + k_3 k_4 z^2}\, z$$

The equations of motion can then be solved by using the first integrals of the system (Cubarsí 1986).

V. PARAMETERS OF THE MODEL

This model has been applied to a sample of 323 luminosity class V, F6 - F7 spectral type stars with residual velocity with respect to Delhaye (1965) local standard of the rest lower than 65 $\mathrm{kms^{-1}}$ taken from a catalogue compiled by Figueras (1986).

The heliocentric velocity of the local standard of the rest and the central momenta up to the fourth order of this sample have been computed by the method given by Núñez & Torra (1982). The heliocentric velocity of the local standard of the rest, opposite to the residual velocity of the Sun, agrees with the value given by Delhaye (1965). It has been obtained that

$$\mu_{\varpi\theta} = \mu_{\theta z} = 0$$

$$\mu_{\varpi\varpi\varpi\theta} = \mu_{\varpi\varpi\theta z} = \mu_{\varpi\theta\theta\theta} = \mu_{\varpi\theta zz} = \mu_{\theta\theta\theta z} = \mu_{\theta zzz} = 0$$

and also that all the third order momenta vanish, as it has been stated they ought to be (Sala, 1986). In table 1 are shown the values of the non zero momenta. From them it can be assumed that

$$\mu_{\varpi z} = \mu_{\varpi\varpi\varpi z} = \mu_{\varpi\theta\theta z} = \mu_{\varpi zzz} = 0$$

and, therefore, $z_{\odot} = 0$. The parameter

$$\rho = \frac{\mu_{\varpi\varpi\varpi\varpi}}{3\mu^2_{\varpi\varpi}} = \frac{\mu_{\varpi\varpi\varpi z}}{3\mu_{\varpi\varpi}\mu_{\varpi z}} = \frac{\mu_{\varpi\varpi\theta\theta}}{\mu_{\varpi\varpi}\mu_{\theta\theta}} = \frac{\mu_{\varpi\varpi zz}}{\mu_{\varpi\varpi}\mu_{zz} + 2\mu^2_{\varpi z}}$$

$$= \frac{\mu_{\varpi\theta\theta z}}{\mu_{\varpi z}\mu_{\theta\theta}} = \frac{\mu_{\varpi zzz}}{3\mu_{\varpi z}\mu_{zz}} = \frac{\mu_{\theta\theta\theta\theta}}{3\mu^2_{\theta\theta}} = \frac{\mu_{\theta\theta zz}}{\mu_{\theta\theta}\mu_{zz}} = \frac{\mu_{zzzz}}{3\mu^2_{zz}}$$

has been determined to be $\rho = 0.92 \pm 0.03$, very close to $\rho = 1$, its value for a Schwarzschild distribution function.

If it is assumed $k_1 = k_3$, and

$$a = \frac{k_2}{k_1} \qquad b = \frac{k_4}{k_1} \qquad \Omega = -\frac{\beta}{k_1}$$

are used, (4.1) will be written, taking $z_{\odot} = 0$,

$$\Pi_0 = -\frac{1}{2}\frac{\dot{b}}{b}\varpi$$

$$\theta_0 = \frac{\Omega\varpi}{1 + a\varpi^2}$$

$$Z_0 = 0$$

Oort constants A, B and C will be then

$$A = \Omega\frac{a\varpi^2}{(1 + a\varpi^2)^2} > 0$$

$$B = -\Omega\frac{1}{(1 + a\varpi^2)^2} < 0$$

$$C = 0$$

and they will satisfy

$$B = -\frac{\mu_{\theta\theta}}{\mu_{\varpi\varpi} - \mu_{\theta\theta}} A$$

TABLE 1. Heliocentric velocity of the local standard of the rest and central momenta up to the fourth order.

U_0	$= -\ 8 \pm 1$	$\mathrm{kms^{-1}}$
V_0	$= -\ 12 \pm 1$	
W_0	$= -\ 7 \pm 1$	
$\mu_{\varpi\varpi}$	$= 598 \pm 41$	$\mathrm{km^2 s^{-2}}$
$\mu_{\varpi z}$	$= -\ 5 \pm 17$	
$\mu_{\theta\theta}$	$= 218 \pm 21$	
μ_{zz}	$= 147 \pm 13$	
$\mu_{\varpi\varpi\varpi\varpi}$	$= 906929 \pm 51117$	$\mathrm{km^4 s^{-4}}$
$\mu_{\varpi\varpi\varpi z}$	$= 1902 \pm 31426$	
$\mu_{\varpi\varpi\theta\theta}$	$= 119648 \pm 14609$	
$\mu_{\varpi\varpi zz}$	$= 98840 \pm 14002$	
$\mu_{\varpi\theta\theta z}$	$= -\ 2409 \pm 7627$	
$\mu_{\varpi zzz}$	$= -\ 1442 \pm 10576$	
$\mu_{\theta\theta\theta\theta}$	$= 184109 \pm 11525$	
$\mu_{\theta\theta zz}$	$= 37387 \pm 6914$	
μ_{zzzz}	$= 74018 \pm 4096$	

The linear approximation of the local velocity field of this sample of stars has been obtained under the hypothesis that their motions are described by the model. The values for the corrections to the precession constants have been determined to be

$$\Delta n = 0.37 \pm 0.25 \ \text{"/century}$$

$$\Delta k = -0.54 \pm 0.26$$

being the latter closer to (Kharchenko et al., 1985)

$$\Delta k = -0.46 \pm 0.16 \ \text{"/century}$$

than to the value of the usual determinations given by Fricke (1977)

$$\Delta n = 0.44 \pm 0.04 \ \text{"/century}$$

$$\Delta k = -0.19 \pm 0.06$$

Oort constants have been found to be

$$A = 16.1 \pm 10.4 \ \mathrm{kms^{-1}kpc^{-1}}$$

$$B = -\ 9.2 \pm 5.6$$

not far from the internationally adopted values

$$A = 15.0 \pm 0.8 \ \mathrm{kms^{-1}kpc^{-1}}$$

$$B = -10.0 \pm 0.8$$

When (Kerr & Lynden-Bell 1985)

$$\bar{\omega}_{\odot} = 8.5 \pm 1.1 \ \mathrm{kpc}$$

is taken to be the solar distance to the galactic center

$$\Pi_0 = 50 \pm 80 \ \mathrm{kms^{-1}} \qquad \frac{\partial \Pi_0}{\partial \bar{\omega}} = 6.2 \pm 9.5 \ \mathrm{kms^{-1}kpc^{-1}} \qquad (5.1)$$

$$\theta_0 = 2.15 \pm 105 \ \mathrm{kms^{-1}} \qquad \frac{\partial \theta_0}{\partial \varpi} = -6.9 \pm 12.0 \ \mathrm{kms^{-1}kpc^{-1}} \qquad (5.2)$$

$$Z_0 = 0 \qquad \frac{\partial Z_0}{\partial z} = 6.2 \pm 9.5 \ \mathrm{kms^{-1}kpc^{-1}}$$

are obtained. The rotation period is then

$$P = 2.4 \times 10^8 \ \mathrm{years}$$

Other parameters of the model have also been determined; tentative values for them are

$$a = 0.024 \ \mathrm{kpc^{-2}} \qquad b = 0.043 \ \mathrm{kpc^{-2}}$$

$$\Omega = 70 \ \mathrm{kms^{-1}kpc^{-1}} \qquad b = -1.72 \times 10^{-17} \mathrm{kpc^{-2}s^{-1}}$$

The curve $\Theta_0(\varpi)$ has a maximum at

$$\bar{\omega}_M = a^{-\frac{1}{2}} = 6.5 \ \mathrm{kpc}$$

with its local standard of the rest moving with a velocity and with a rotation period given by

$$\Pi_M = 40 \ \mathrm{kms^{-1}} \qquad P_M = 1.8 \times 10^8 \ \mathrm{years}$$

$$\theta_M = 225 \ \mathrm{kms^{-1}}$$

respectively. If Oort (1965) value

$$k^2 = 0.9 \times 10^{-29} s^{-2}$$

is adopted, the period of the motion of the stars will be

$$P = \frac{2\pi}{k} = 0.6 \times 10^8 \text{ years}$$

Through hydrodynamic equation 100 (Sala et al. 1985) and assuming (Stodólkiewicz 1973)

$$\frac{\partial \ln N}{\partial \varpi} = -0.1 \text{ kpc}^{-1}$$

the radial force will be

$$-\frac{\partial U}{\partial \varpi} = \frac{2B_1}{\varpi^3} - k^2\varpi > -78000 \text{ km}^2 s^{-2} \text{kpc}^{-1}$$

where

$$\frac{2B_1}{\varpi^3} > -5000 \text{ km}^2 s^{-2} \text{kpc}^{-1} \qquad -k^2\varpi = -73000 \text{ km}^2 s^{-2} \text{kpc}^{-1}$$

From it, the potential U will be evaluated to be

$$U = \frac{B_1}{\varpi^2} + \frac{k^2}{2}\varpi^2 > 289000 \text{ km}^2 s^{-2}$$

with

$$\frac{B_1}{\varpi^2} > -20000 \text{ km}^2 s^{-2} \qquad \frac{k^2}{2}\varpi^2 = 309000 \text{ km}^2 s^{-2}$$

Table 2 contains some of the values not even determined under the assumption that a star with velocity given by (5.1) and (5.2) reaches an apocenter and a pericenter

$$\varpi_A = 10.5 \text{ kpc}$$

$$\varpi_P = 6 \quad \text{kpc}$$

TABLE 2. Parameters of the model under the hypothesis that the path of a star with velocity $\Pi = 50$ kms^{-1} and $\theta - 215$ kms^{-1} has an apocenter $\varpi_A = 10.5$ kpc and a pericenter $\varpi_P = 6$ kpc.

$\frac{\ddot{k}_1}{k_1}$	$= -0.42 \times 10^{-29} s^{-2}$
$\frac{\partial \Pi_0}{\partial t}$	$= -1.9 \times 10^{-29}$ kpcs^{-2}
	$= 18000$ km^{2}s^{-2}kpc^{-1}
$-\frac{\partial U}{\partial \varpi}$	$= -23000$ km^{2}s^{-2}kpc^{-1}
$\frac{2B_1}{\varpi^3}$	$= -50000$ km^{2}s^{-2}kpc^{-1}
B_1	$= 1.5 \times 10^{-26}$ kpc^{4}s^{-1}
θ_C	$= 440$ kms^{-1}
$\frac{B_1}{\varpi^2}$	$= 210900$ km^{2}s^{-2}
U	$= 515900$ km^{2}s^{-2}
I_1	$= 1082700$ km^{2}s^{-2}
L	$= 6.2$
P_L	$= 1$ rad

VI. SUMMARY

An axi-symmetric time depending kinematic galactic model with separable potential has been derived under the hypothesis of Chandrasekhar. The properties of the obtained potential, given by (3.1), have been described, and with the aid of the first integrals of the system, the motion of the stars has been determined.

Within the model, the central momenta up to the fourth order and a linear approximation of the velocity field have been obtained for a sample of luminosity class V, F6 - F7 spectral type stars. The obtained values, that fairly agree with the determinations of other authors, show the validity of the model to give account of some observational properties of galactic kinematics. The parameters of the model have been determined, and it must be pointed out the importance of the galactic halo in the obtained value of the galactic potential.

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DISCUSSION

PIŞMIŞ: Se sabe que las dos primeras integrales son aislantes, sospecho que las integrales 3 y 4 no lo son ¿no es así?

SALA: Las integrales tercera y cuarta son localmente aislantes.

AGUILAR: ¿Cuál es la forma funcional asumida para la distribución en el espacio fase?

SALA: La función es del tipo Schwarzschild generalizado.

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