SOME INSTABILITIES IN THE 3/1 RESONANT PROBLEM

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RESUMO. Usando o método de Laplace-Lagrange, é obtido um critério bastante simples de instabilidade para o problema ressonante 3/1 plano.

ABSTRACT. By using Laplace-Lagrange method, a very simple criterion of instability is obtained for the 3/1 planar resonance problem.

Key words: ASTEROIDS — INSTABILITIES

I. INTRODUCTION

The stability of the asteroids in the cases of resonances has been studied by several authors under many points of view.

Recently, possible mechanisms for the formation of the Kirkwood gaps and discussion of relevant papers in this area, has been reviewed by Scholl (1985).

In this paper, the classical method of Laplace-Lagrange is used to study a simplified model of 3:1 resonance. It is shown that the resulting equations present three types of solutions, two of them are unstable. These unstable solutions are in fact situated into the observed 3:1 Kirkwood gap, however the obtained width of instability is still too narrow. Actually the question is whether or not this can be applied for other resonances and also to investigate if this instability can generate the Kirkwood gaps.

II. THE PROBLEM

After elimination of short periodic terms, the hamiltonian for this problem, in the planar case is given by (Yokoyama and Sato 1986)

$$F = \frac{\mu^2}{2L^2} - n'\Lambda + M_1 e^2 + T_1 e\, e' \cos(\tilde{\omega} - \tilde{\omega}') + M_3 e^2 \cos(3\lambda' - \lambda - 2\tilde{\omega}) +$$

$$+ M_5 e\, e' \cos(3\lambda' - \lambda - \tilde{\omega} - \tilde{\omega}') + O(e^3, e'^2) \tag{2.1}$$

Where Λ is the conjugated momentum related to λ' and T_1, M_1, M_3, M_5 are functions of a/a', which in this case will be taken to be equal to 0.4805968.

By using the following set of canonical variables:

$$\alpha_1 = \lambda - 3\lambda' \qquad\qquad A_1 = L$$

$$q = -[2(L-G)]^{1/2} \sin\tilde{\omega} \qquad p = [2(L-G)]^{1/2} \cos\tilde{\omega}$$

and expanding (2.1) in the neighborhood of the exact resonant point $A_{10} = (\mu^2/3n')^{1/3}$, we have:

$$F = \frac{\mu^2}{2A_{10}^4} x^2 + \frac{M_{10}}{A_{10}} (p^2 + q^2) + \frac{T_{10} e'}{\sqrt{A_{10}}} [p \cos\tilde{\omega}' - q \sin\tilde{\omega}'] +$$

$$+ \frac{M_{30}}{A_{10}} \left[(p^2 - q^2) \cos \alpha_1 + 2pq \sin \alpha_1 \right] + \frac{M_{50}}{\sqrt{A_{10}}} e' \left[p \cos(\alpha_1 + \tilde{\omega}') + \right.$$

$$\left. + q \sin(\alpha_1 + \tilde{\omega}') \right] + 0 \left(x^3, T_{10} x, M_{10} x \right) \qquad (2.2)$$

Where $x = (A_1 - A_{10})$

The equations of motion are:

$$\frac{dp}{dt} = \frac{\partial F}{\partial q} \qquad\qquad \frac{dq}{dt} = - \frac{\partial F}{\partial p} \qquad\qquad \frac{d^2 \alpha_1}{dt^2} = - \frac{3\mu^2}{A_{10}^4} \frac{\partial F}{\partial \alpha_1} \qquad (2.3)$$

In order to study the behaviour of (p,q), Koiller et al. (1986) assumed the "constrained system" by imposing in the first approximation:

$$\partial F / \partial \alpha_1 = 0$$
$$\qquad (2.4)$$

Now, if (2.4) is solved for $\alpha_1 = \alpha_1 (p,q)$ and substituting this in the first two equations of (2.3), it results a Hamiltonian system in (p,q) variables only, and study can go ahead.

However, this technique of decoupling system (2.3) can be interpreted in a classical way which leads to the Laplace-Lagrange method: if (2.4) holds, by the last equation of (2.3) it implies that α_1 is a linear function of time. This means that, in first order, the mean motion σ_0 of the angle α_1 is kept constant, while the first two equations in (2.3) become linear equations with periodic coefficients. Therefore, the next step is to find $p(t)$, $q(t)$.

III. TYPES OF SOLUTIONS

By taking a convenient periodic linear transformation, (Liapunov transformation) it is possible to reduce the system (p,q) into a linear with constant coefficients.

Actually, we have found this transformation and obtained the complete solutions for (p,q). Then it is shown that they are stable (quasi-periodic) if:

$$\Delta = \left(\frac{4M_{10}}{A_{10}} + \sigma_0 \right)^2 - \frac{64 M_{30}^2}{A_{10}^2} > 0 \qquad \text{and}$$

$$\Delta \neq \sigma_0^2 \qquad\qquad (3.1)$$

Solutions have a linear term in t if:

$$\Delta = 0 \qquad \text{or}$$

$$\Delta = \sigma_0^2 \qquad\qquad (3.2)$$

and exponentially increasing terms if:

$$\Delta < 0 \qquad\qquad (3.3)$$

Conditions (3.2) give the bounds of σ_0 such that instabilities occur. In terms of semi-major axes we found:

$$\Delta \leqslant 0 \qquad \text{for} \qquad 2.498\ 961 \quad \leqslant \quad a \quad \leqslant \quad 2.502\ 622$$

$$\Delta = \sigma_0^2 \qquad \text{for} \qquad a = 2.496\ 728$$

Indeed, this width is too narrow when compared with observed 3:1 Kirkwood gap, however, even outside of this interval again, eccentricity can attain high (catastrophic) values provided a is near to one of the extrema of this interval.

The full calculations about the critical values for the ends of this interval are being undertaken and we hope to pursue in a next paper.
We also applied this study to the 2:1 resonance and, preliminar results, show the same type of instabilities.
Final results including other resonances, and details about Liapunov transformation, as well as, explicit solutions for (p,q) should be presented in a forthcoming paper.

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