. Mexicana Astron. Astrof., 21, 251-254 (1990)

ANALYTIC TIME DEPENDING GALAXY MODELS

Ferran Sala

Departament de Física de l'Atmosfera, Astronomia i Astrofísica
Universitat de Barcelona

RESUMEN. Considerando las hipótesis de Chandrasekhar para el estudio de la Dinámica Galáctica, se han desarrollado varios modelos galácticos analíticos integrables con simetría axial y dependientes del tiempo.

ABSTRACT. By considering Chandrasekhar hypotheses for the study of Galactic Dynamics, several integrable analytic axisymmetric time-depending galactic models have been developed.

Key words; GALAXY-DYNAMICS — GALAXY-STRUCTURE

I. INTRODUCTION

Wide regions of the Galaxy can be studied as a dynamical system whose distribution function f in phase space satisfies the collisionless Boltzmann equation

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \mathbf{V}\cdot\nabla_{\mathbf{r}} f - \nabla_{\mathbf{r}} U\cdot\nabla_{\mathbf{V}} f = 0 \qquad (1.1)$$

where t is the time, $\mathbf{r}$ and $\mathbf{V}$ are the position and the velocity of a star and U is the potential per unit mass.

Chandrasekhar (1942) developed a theory of the Galactic Dynamics based on three fundamental hypotheses:
1) At any point $\mathbf{r}$ of the system it is possible to define a local standard of rest, whose velocity is $\mathbf{V}_0(t,\mathbf{r})$.
2) The distribution function f is of the generalized Schwarzschild type

$$f(t,\mathbf{r},\mathbf{V}) = f(Q + \sigma) \qquad (1.2)$$

where Q is a quadratic form

$$Q = \mathbf{v}^t\cdot\mathbb{A}\cdot\mathbf{v}$$

being

$$\mathbf{v} = \mathbf{V} - \mathbf{V}_0$$

the residual or peculiar velocity of a star, $\mathbb{A}(t,\mathbf{r})$ a symmetric second order tensor and $\sigma(t,\mathbf{r})$ a scalar function.
3) The motions of stars are governed by the potential $U(t,\mathbf{r})$ per unit mass.

The fundamental equation (1.1) admits a solution of the form (1.2) when

$$\frac{D(Q + \sigma)}{Dt} = 0 \qquad (1.3)$$

If

$$\boldsymbol{\Delta} = \mathbb{A}\cdot\mathbf{V}_0$$

$$-\chi = \mathbf{V}_0{}^t\cdot\mathbb{A}\cdot\mathbf{V}_0 + \sigma = \boldsymbol{\Delta}\cdot\mathbf{V}_0 + \sigma$$

are introduced, (1.3) can be expressed (Orús 1952) by the four tensor equations

$$\begin{aligned} &\mathbf{def}\ \mathbb{A} = 0 \\ &\mathbf{def}\ \boldsymbol{\Delta} = \frac{1}{2}\,\frac{\partial\mathbb{A}}{\partial t} \end{aligned} \qquad (1.4)$$

$$\mathbb{A}\cdot\nabla U + \frac{\partial\Delta}{\partial t} = -\frac{1}{2}\nabla\chi$$
$$\Delta\cdot\nabla U = \frac{1}{2}\frac{\partial\chi}{\partial t} \qquad (1.5)$$

where **def** denotes a generalization of the strain operator of the ellasticity theory (Orús 1952).

II. DERIVATION OF THE MODELS

If the system is assumed to be symmetric with respect to the equatorial plane and to the rotation axis, by solving the equations (1.4) in cylindrical coordinates (ϖ,ϕ,z) it is obtained (Sala 1986)

$$A_{\varpi\varpi} = k_1 + k_4 z^2 \qquad A_{\phi\phi} = k_1 + k_2\varpi^2 + k_4 z^2 \qquad \Delta_\varpi = \frac{1}{2}k_1'\varpi$$
$$A_{\varpi\phi} = 0 \qquad A_{\phi z} = 0 \qquad \Delta_\phi = -\beta\varpi$$
$$A_{\varpi z} = -k_4\varpi z \qquad A_{zz} = k_3 + k_4\varpi^2 \qquad \Delta_z = \frac{1}{2}k_3' z$$

where k_1 and k_3 are positive functions of the time t, k_2 and k_4 are positive constants and β is a negative constant provided that the azimuth angle ϕ increases with the rotation of the Galaxy.

The solution of the equations (1.5) allows for the determination of the potential U and the function χ. A general description of its derivation can be found in Sala (1989). Three different types of solutions are found depending on the relations between the functions k_1 and k_3.

Case 1: $k_1 = k_3$. The solutions can be written in spherical coordinates (r,ϕ,θ). They are

$$U = \left[-\frac{k_1''}{4} + \frac{k_1'^2}{8k_1}\right]\frac{r^2}{k_1} + \frac{1}{k_1}\left[U_1\left(\frac{r^2}{k_1}\right) + \frac{k_1}{r^2}U_2(\theta)\right]$$
$$-\frac{1}{2}\chi = \frac{k_1'^2}{8}\frac{r^2}{k_1} + U_1\left(\frac{r^2}{k_1}\right) + \left[\frac{k_1}{r^2} + k_4\right]U_2(\theta) + \text{constant} \qquad (2.1)$$

where U_1 and U_2 are arbitrary functions.

The first integrals of the system are

$$I_1 \equiv k_1\left(\Pi - \frac{1}{2}\frac{k_1'}{k_1}\varpi\right)^2 + k_1\Phi^2 + k_1\left(Z - \frac{1}{2}\frac{k_1'}{k_1}z\right)^2$$
$$+ 2U_1\left(\frac{r^2}{k_1}\right) + \frac{2k_1}{r^2}U_2(\theta) = \text{constant}$$
$$I_2 \equiv \varpi\Phi = \text{constant} \qquad (2.2)$$
$$I_3 \equiv (z\Pi - \varpi Z)^2 + z^2\Phi^2 + 2U_2(\theta) = \text{constant}$$

where (Π,Φ,Z) are the physical cylindrical components of the velocity of a star.

Case 2: $k_1 \neq k_3$, but $k_1'k_3 = k_3'k_1$. The solutions can be written in prolate spheroidal coordinates (λ,ϕ,ν) defined by

$$\left.\begin{matrix}\lambda\\ \nu\end{matrix}\right\} = \frac{1}{2}\left\{-\alpha-\gamma+\varpi^2+z^2 \pm [(-\alpha-\gamma+\varpi^2+z^2)^2 - 4(\alpha\gamma-\gamma\varpi^2-\alpha z^2)]^{1/2}\right\}$$

where $\alpha = -k_3/k_4$ and $\gamma = -k_1/k_4$. They are

$$U = \left[-\frac{k_1''-k_3''}{4} + \frac{(k_1'-k_3')^2}{8(k_1-k_3)}\right]\frac{\lambda+\nu}{k_1-k_3}$$
$$+ \frac{1}{k_1-k_3}\left[\frac{F(\lambda/(k_1-k_3)) - F(\nu/(k_1-k_3))}{(\lambda-\nu)\,/\,(k_1-k_3)}\right] \qquad (2.3)$$
$$-\frac{1}{2}\chi = k_4\left\{-\frac{k_1'k_3'}{8k_4^2} + \frac{(k_1'-k_3')^2}{8}\frac{\lambda\nu}{(k_1-k_3)^2}\right.$$
$$\left.+ \frac{(\nu/(k_1-k_3))\,F(\lambda/(k_1-k_3)) - (\lambda/(k_1-k_3))\,F(\nu/(k_1-k_3))}{(\lambda-\nu)\,/\,(k_1-k_3)}\right\} + \text{constant}$$

where F is an arbitrary function.

The first integrals of the system are

$$I_1 \equiv k_1(\Pi - \frac{1}{2}\frac{k_1'}{k_1}\varpi)^2 + k_1\Phi^2 + k_1(Z - \frac{1}{2}\frac{k_1'}{k_1}z)^2 + 2k_1\frac{F(\lambda/(k_1-k_3)) - F(\nu/(k_1-k_3))}{\lambda - \nu} = \text{constant}$$

$$I_2 \equiv \varpi\Phi = \text{constant} \tag{2.4}$$

$$I_3 \equiv (z\Pi - \varpi Z)^2 + z^2\Phi^2 - \frac{k_1-k_3}{k_4}(Z - \frac{1}{2}\frac{k_1'-k_3'}{k_1-k_3}z)^2 - \frac{2}{\lambda - \nu}[((k_1/k_4)-\nu)\ F(\lambda/(k_1-k_3)) - ((k_1/k_4)-\lambda)\ F(\nu/(k_1-k_3))] = \text{constant}$$

Case 3. $k_1'k_3 \neq k_3'k_1$. The solutions have already been described (Sala 1987).

The sets of first integrals (2.2) and (2.4) are in involution, so that, the solutions found in the cases 1 and 2 determine integrable systems, being the equations of motion solved as quadratures.

III. KINEMATICAL AND DYNAMICAL CONSEQUENCES.

In the cases 1 and 2, the physical cylindrical components of the velocity of the local standard of rest are

$$\Pi_0 = \frac{1}{2}\frac{k_1'}{k_1}\varpi$$

$$\Phi_0 = -\frac{\beta\varpi}{k_1+k_2\varpi^2+k_4z^2}$$

$$Z_0 = \frac{1}{2}\frac{k_3'}{k_3}z$$

while its motion is given by

$$\varpi = c_\varpi\psi$$

$$\phi - \phi_0 = -\frac{\beta}{\varkappa_1+k_2c_\varpi^2+k_4c_z^2}\int\frac{dt}{\psi^2}$$

$$z = c_z\psi$$

where c_ϖ, ϕ_0 and c_z are constants and ψ is a function of time t defined by

$$k_1 = \varkappa_1\psi^2$$
$$k_3 = \varkappa_3\psi^2 \tag{3.1}$$

where $\varkappa_1$ and $\varkappa_3$ are constants, being $\varkappa_1 = \varkappa_3$ in the case 1.

If a new cylindrical coordinate system (c_ϖ, ϕ, c_z) and a new time (Chandrasekhar, 1942)

$$\tau = \int\frac{dt}{\psi^2} \tag{3.2}$$

are introduced, the motion of the local standard of rest is reduced to the circular differential rotation found in the steady state systems.

In the case 1, defining a new spherical coordinate system (c_r, ϕ, θ) where

$$r = c_r\psi$$

being ψ given by (3.1), and with the new time τ defined in (3.2), the potential (2.1) can now be written as

$$U = V_1(c_r^2) + \frac{1}{c_r^2}V_2(\theta) \tag{3.3}$$

where V_1 and V_2 are arbitrary functions.

In the case 2, defining a new prolate spheroidal system $(c_\lambda, \phi, c_\nu$ where

$$\lambda = c_\lambda \psi^2$$

$$\nu = c_\nu \psi^2$$

and with the new time τ, the potential (2.3) can now be expressed by

$$U = \frac{G(c_\lambda) - G(c_\nu)}{c_\lambda - c_\nu} \qquad (3.4)$$

where G is an arbitrary function.

The potentials (3.3) and (3.4) have the forms of those found in the search of steady state solutions of the equations (1.5). The first integrals in (2.2) and (2.4) are reduced, in the new coordinates, to the energy, the angular momentum and the classical third integrals.

Being in the transformed coordinates the forms of the potentials those of the steady state potentials, the classification of the orbits in the transformed coordinates is that obtained by de Zeeuw (1985). Bound angular momentum orbits are stable short axis tubes limited by coordinate surfaces or derived special cases, being possible both direct and retrograde orbits.

IV. SUMMARY AND CONCLUSIONS

The most general non stationary solutions of the Chandrasekhar theory for rotating systems with an axis and a plane of symmetry have been found. The solutions found in the cases 1 and 2 describe dynamical systems with three degrees of freedom and three isolating integrals in involution, so that they are integrable. Transformed space and time coordinates can be introduced, so that the orbit structure can be accounted for.

Acknowledgement: The author wishes to thank Dr. M. Sanromà for his collaboration in making possible this contribution to the VI Latin American Regional Meeting of the International Astronomical Union.

REFERENCES

Chandrasekhar, S. 1942, *Principles of Stellar Dynamics* (Chicago: University of Chicago Press)
Orús, J.J. de 1952, *Collectanea Mathematica*, vol. v
Sala, F. 1986, *Ph. D. Thesis*, University of Barcelona
Sala, F. 1987, *Rev. Mex. Astron. Astrof.* **14**, 195
Sala, F. 1989, *Astron. Astrophys.* (submitted)
de Zeeuw, T. 1985, *Monthly Notices Roy. Astron. Soc.* **216**, 273

Ferran Sala: Departament de Física de l'Atmosfera, Astronomia i Astrofísica, Universitat de Barcelona, Av. Diagonal, 647, 08028 Barcelona, Spain.