Rev. Mexicana Astron. Astrof., **21**, 581-584 (1990)

ON THE DETERMINATION OF PERTURBATIONS IN PLANETARY MOTIONS

Adrian Brunini
CONICET, Argentina

ABSTRACT. The main goal of this paper is to give an alternative way when the physical reality is too difficult to interpret, in such form that small effects can be ignored in the force model of dynamical systems.

RESUMEN. El principal objetivo de este trabajo es ofrecer una vía alterna para cuando la realidad física es demasiado difícil de interpretar, en tal forma que los efectos pequeños puedan despreciarse en los modelos de fuerza de sistemas dinámicos.

Key words: PLANETS AND SATELLITES-DYNAMICS

INTRODUCTION.

The aim of this work is to determine perturbing functions affecting ordinary Differential Equations Systems, in order to get solutions of them that approximate a set of given measurements.

A mathematical model for these perturbations not always can be completely formulated. In this situation the parameter identification of a model of the perturbation by means of statistical methods can not be used.

Recently, Zadunaisky (1988, 1988a, 1988b) and Rodriguez (1987, 1988) developed a method to emulate such pertubations without model, obtaining very efficient numerical results for problems where the measurable data consist of values of the system solution, that may be affected by random errors.

This is not indeed the situation in practice provided the measures are nonlinear functions of the system solution (It is well known that in Astronomy, measurable data are meanly angles, times, distances...).

In the present work, we present a method which makes possible to estimate those pertubations in the general case of indirect measurements. Also, we present the main results ooncerning of error bounds. Finally, in the last section , we shall describe numerical results of the application of this method to a problem related with the planetary motion.

ESTIMATION ALGORITHM

Although this method is applicable to a set of ODEs, for the sake of simplicity we shall consider an Initial-Value problem with a single equation of the form:

$$y'(t)=f(t,,y(t)+P(t)...\quad y(t_o)=y_o \qquad (1)$$

Where P(t) is an unknown perturbation that depends on t. We wish to compute this perturbation numerically in order to obtain a solution y(t), of the I.V.P. (1) which fits a given data set. Let us consider the case of indirect measures of the form

$$\alpha_n=\alpha(y(t_n))+\delta_n \qquad (2)$$

Where δ are measurement errors.

Now, let us define a "*reference problem*" as:

$$z'(t)=f(t,z(t))+P_n \qquad (3)$$

that we have obtained from eq.(1) by replacing the unknown perturbation p(t) by a constant value Pn. To obtain estimates of P(t) in the instant Tn we adjust an initial value z(tn) and a constant Pn so that the solution of eq.(3) satisfies the boundary conditions in t_n and t_{n+j}. To carry out this procedure we can use any version of the Simple Shooting Method (Isaakson and Keller 1988).

ERROR BOUND AND OPTIMAL SCHEMES

The conditions that must be fullfiled by the functions involved in order to prove the existence and unicity of the solution of the problem can be seen in Brunini (1988), as well as a complete numerioal analysis of the estimation.

However, in order to complete this presentation we wish show the main results concerning the error bounds in the estimation.

If we assume that P and f are sufficiently regular functions, and we particularly suppose that the function α has at least continuous first derivatives, may be proved that exist two values of S and P_n such the measurements are satisfied, and the principal term in the error expansion is:

$$P(t_n)-P_n = 2\Delta/h+P'(t_n)jh/2 \qquad (4)$$

Where

$$\Delta=max\ \{|\delta_{n+j}\ \partial y(t_{n+j})/\partial\alpha|,|\ \delta_n\ \partial y(t_n)/\partial\alpha|\}$$

eq (4) show that the error in the estimation of P(tn) have two principal components. However, the computational evidence shows that the principal source of error is arising from the measurements errors, that is magnified by a factor 1/h.

Let us remark that if we would apply the ideas developed in the previous work by Zadunaisky (1988) and Rodriguez (1988), where to make possible to estimate the perturbation must be previously obtained values of the solution y(t) of the system, by solving the boundary value problem defined by eq. (3) subject to the boundary conditions (2). The $Y(t_n)$ obtained in this way, are affected by the unknown perturbation, dropped in the ODEs justly because the mathematical model for P(t) is unknown. Clearly, we can not use these values to estimate the perturbation because of an obvious circularity.

NUMERICAL RESULTS

To prove this method, we tried to determine perturbing effects of a massive body on other "obervable" ones. We intend to estimate the perturbation caused by Neptune on Saturn and Uranus. The first step was the simulation of

Ieliocentric Equatorial coordinates for the known planets. The computational nethod followed was a high precision numerioal integration of the three planet system. The ODEs of motion, in an Heliocentric sytem are well known, but we pelieve that it is necessary to note that we intend to simulate a "real situation". Then, numerical values for the dynamical parameters used in this stage of the work, must not be the same ones that we shall use in the stage of estimation, because in general, available numerical values for masses and initial conditions of perturbed planets are affected by the perturbation that nas not been considered in the dynamical model when these ones have been determined.

All the parameters such as masses and initial conditions for the three planets, have been taken from the work of Osterwinter and Cohen (1972). In this way we have computed Heliocentric rectangular equatorial coordinates for the 3 planets, spaced 100 days from J.D. 2446400 (1-12-1985) to J.D. 2451600 (25-2-2000). The closest approach of the 3 planets occurs just in the centre of the interval, where the unknown outer planet will have its greatest effect upon both known ones.

The next step was the calculation of the angular coordinates for Saturn and Uranus. In all cases. simulated measurements have heen caloulated, adding to these values, "measurement" errors taken from a zero mean normal random sequence with a realistic variance taken from the same paper of Oesterwinter and Cohen.

We are now in a position to investigate the accuracy of our numerical scheme. We must determine 18 unknowns: 8 initial conditions and 3 perturbation for each planet. We must use measures in 5 instants for each estimation, taken into account that the angular measures are 4 in each one.

Since the previous numerical evidence have shown that the principal source of error is arising from the measurements, small stepsizes can not be used. The previous experience by Zadunaiskv and Rodriguez suggest that the optimal stepsize is between 500 and 2000 days. So, we have used for the estimation a stepsize of 500 days, and however each estimation is performed covering a whole interval of 4000 days.

In this way, we may obtain 12 estimates of P(t), centered at J.D. 2449000 and spaced 100 days.

The results are summarized in the Table I. The efficiency of each computed magnitude of Pn has been estimated by means of tne expression

$$\mathrm{eff}(P)=-\log_{10} \quad (P(t_n)-P_n)/P(t_n)$$

When this efficiency is positive it roughly counts the number of significant decimals correctly estimated.

Our results look reasonably good. In all cases we have at least estimated the order of magnitude of the perturbation. This is because the motions of the planets are slow, so the truncation error, that depends on derivatives of P(t) is small in comparison to the component arising to the measurement errors. In order to prove this fact, we have tryed different estimations with an increasing sequence of stepsizes, and we have found that the truncation errors become evident from stepsizes of 1200 days.

CONCLUDING REMARKS

The results obtained under simulated -but realistic- conditions indicate that the proposed estimation scheme can be considered as a candidate to be used in real problems where the force models are known only approximately.

TABLE I. Accuracy in the estimation of P(t)

J.D. 244+	P_u (x1000)	eff.	P_s (x1000)	eff
8400	.1515	1.23	.2515	0.62
8500	.1531	1.46	.2438	0.71
8600	.1545	1.44	.2353	0.70
8700	.1559	1.33	.2263	0.67
8800	.1570	1.32	.2169	0.64
8900	.1582	1.41	.2073	0.59
9000	.1590	1.42	.1975	0.52
9100	.1597	1.59	.1875	0.62
9200	.1607	1.37	.1776	0.50
9300	.1607	1.34	.1678	0.58
9400	.1608	1.40	.1581	0.63
9500	.1613	1.32	.1487	0.55

In such real cases our method may help to build more properly a model of the physical phenomena.

We believe that this method is a powerful tool to attempt the discovery of an hypothetical ten planet, that is actually conjectured from the irregularities of the observed path of Neptune in the sky.

To conclude, we believe that the method may be generalized to other types of problems not necessarily related with the field of differential equations.

REFERENCES

Brunini A.: Ph. D. Thesis, Universidad Nacional de la Plata (1988)

Isaakson E. and Keller H.B.: *Analysis of Numerical Methods*. John Wiley and Sons, Inc (1966)

Oesterwinter C. and Cohen C.J.: *Cel, Mech.*, 15 (1972)

Rawlins D. and Hammerton M.: *Mon. Not. Roy. Astr. Soc.*, 162 (1973)

Rodriguez R.: Ph. D. Thesis, Universidad de la Plata

Rodriguez R.: submitted

Zadunaisky P.E.: *Proc. of the Workshop Numerical Treatment of Inverse Problem in Differential and Integral Equations*, (P. Deuflhard ed.) pp 62-72.

Zadunaisky P.E. and Sanchez Pena R.S.: *Journal of Guidance, Control and Dynamics, A.I.A.A.* (to appear)

Adrian Brunini : Facultad de Ciencias Astronómicas y Geofísicas, Universidad Nacional de La Plata (1900) La Plata, Argentina