ev. Mexicana Astron. Astrof., **24**, 113-120 (1992)

# DYNAMICAL CALCULATIONS IN THE HYDRODYNAMICAL APPROACH OF A PLASMA-PHOTON GAS MODEL OF THE UNIVERSE IN THE RECOMBINATION ERA

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*Received 1991 August 27*

RESUMEN

Utilizando métodos de integración numérica se resuelven las ecuaciones de movimiento dinámicas que caracterizan un modelo de Universo constituido de dos fluidos durante la era de la recombinación. Asumiendo perturbaciones iniciales en la densidad de materia del orden de $10^{-6}$ y en la densidad de radiación de $10^{-12}$ se obtienen solamente soluciones decrecientes.

ABSTRACT

Via numerical methods, the dynamic equations of motion for characterizing a two-component fluid model of the Universe during the recombination era are solved. The solutions show only time decreasing behavior for initial conditions in matter, and radiation, density perturbations of the order of $10^{-6}$ and $10^{-12}$ respectively.

*Key words:* **COSMOLOGY**

## I. INTRODUCTION

Corona (1987) has analyzed the equations of motion for characterizing cosmological models in the ynchronous gauge and has given static solutions for a three-component model of the Universe during the ecombination era. A static solution for a two-component fluid model, has been worked out by Nowotny 981) and the dynamic treatment for one-component fluids has been carried out by many authors, (e.g. ifshitz 1946; Lifshitz & Khalatnikov 1973; Peebles 1968; Weinberg 1972). However no dynamic solution or the two-component fluid model has been produced, and the purpose of this paper is to deal with this roblem. This model leads to differential equations with non-constant coefficients. Therefore, the *ansatz* $e^{wt}$ or the temporal behavior of the solution is no longer appropriate. We have to resort to a numerical solution f a coupled system of ordinary differential equations for the amplitudes of the matter density $\tilde{A}(t)$, radiation ensity $\tilde{T}(t)$, perturbation of the metric $\tilde{H}(t)$, and of its derivative $\tilde{G}(t)$, and perturbations on velocities of the atter fluid $\tilde{\xi}(t)$, and radiation fluid $\tilde{\zeta}(t)$, respectively.

Because of numerical difficulties, we will consider the Universe only as a binary photon-gas mixture with ompton interactions. We will assume for the gas an equation of state of the form

$$p_g = (1 + x)(\rho_g/m_H)k_B T , \tag{1}$$

ith the following caloric equation

$$\varepsilon_g = p_g/(\gamma - 1) + (\chi/m_H)\rho_g x \tag{2}$$

here $x$ is the degree of ionization, $\chi$ is the ionization potential of a hydrogen atom, $\gamma$ is the adiabatic index ($\gamma$ = 5/3 for a mono-atomic gas), $p_g$ is the pressure of the fluid, $\rho_g$ the density, and $m_H$ is the mass of a hydrogen tom.

A very difficult problem for dynamic calculations, is to find a suitable set of initial conditions for tl amplitudes of the perturbations (see Rose & Corona-Galindo 1991). However, we have solved the equatio for $\delta\rho_g/\rho_{go} \approx 10^{-6}$ and $\delta\rho_r/\rho_{ro} \approx 10^{-12}$ and we have obtained only decreasing modes. Although no unstab mode is yet obtained, we consider of interest to report the method of integration of the equations.

The distribution of the paper is the following: in §II we write the equations of motion, in §III we integra Friedmann equation, in §IV we solve the equations of motion and in §V we discuss the results. Througho the paper the greek indexes run from 0 to 3, latin indexes run from 1 to 3.

## II. EQUATIONS OF MOTION FOR THE FLUIDS

The set of linearized equations that describe this model in a comoving coordinate system were obtain by Corona (1987). For this reason we omit here their derivation and we can immediately write down tl equations that characterize the system:

The momentum balance equations for the gas and radiation are

$$\rho_{go}\,\delta\dot{v}_g^i + 2H\rho_{go}\,\delta v_g^i + (\delta p_{g,i}/a^2) - (4/3)(\rho_{ro}/\tau_o)(\delta v_r^i - \delta v_g^i) = 0\,, \quad (\text{[truncated]}$$

$$\rho_{ro}\,\delta\dot{v}_r^i + H\,\rho_{ro}\delta\, v_r^i + (3c_s^2/4a^2)\,\delta\rho_{r,i} + (\rho_{ro}/\tau_o)(\delta\, v_r^i - \delta\, v_g^i) = 0\,. \quad (\text{[truncated]}$$

The continuity equation of the gas and the energy equation of the photon fluid are

$$\delta\dot{\rho}_g + 3H\,\delta\rho_g + \rho_{go}\,\delta\, v_{g,i}^i + (1/2a^2)\,\rho_{go}\,\dot{h}_{ii} - H\,(\rho_{go}/a^2)\,h_{ii} = 0\,, \quad (\text{[truncated]}$$

$$\delta\dot{\rho}_r + 4H\,\delta\rho_r + (4/3)\,\rho_{ro}\,\delta v_{r,i}^i + (2/3)\,(\rho_{ro}/a^2)\,\dot{h}_{ii} = 0\,. \quad (\text{[truncated]}$$

The energy equation for the gas-component is

$$\delta\dot{p}_g + 3H\gamma\,\delta p_g + (\gamma - 1)(\varepsilon_{go} + p_{go})\,\delta v_{g,i}^i + $$
$$+ (1/2a^2)(\varepsilon_{go} + p_{go})(\gamma - 1)\,\dot{h}_{ii} = 0\,. \quad (\text{[truncated]}$$

The gravitational field equation is

$$\ddot{h}_{ii} + 2H\,\dot{h}_{ii} = -8\pi G(\delta\rho_g + 2\,\delta\rho_r)\,. \quad (\text{[truncated]}$$

As already mentioned, the coefficients of the system of equations (3)-(8) are now time-dependent. Findin analaytic solutions is very difficult, however, a numerical solution is possible and in this section we will repo a series of solutions for particular initial conditions.

With the assumption that both radiation and gas are in local thermodynamical equilibrium, the model characterized by only five equations. For this reason, the equation of energy for the gas will be left aside, an with the aid of the equation of state for the gas, the pressure in equation (3) can be eliminated.

Actually, according to Eq. (1), a perturbation in pressure gives

$$\delta p_g = (1 + x)(k_B\,T_o/m_H)\,\delta\rho_g + (1 + x)(k_B\,T_o\,\rho_{go}/4\,m_H\,\rho_{ro})\,\delta\rho_r + $$
$$+ (k_B\,T_o\,\rho_{go}/m_H)\,\delta x\,, \quad (\text{[truncated]}$$

where the relation $\rho_r = (4\sigma T^4)/c$ between the radiation density and temperature has been used.

The degree of ionization is obtained from the Saha equation for a pure hydrogen gas

$$x^2/(1 - x) = (2\pi\, m_e\, k_B\, T_o)^{3/2}\,(m_H/h^3\,\rho_{go})exp\,(-\chi/k_B\,T_o)\,, \quad (1\text{[truncated]}$$

$x = n_e/n$ is the degree of ionization, $n_e$ and $n$ are the number densities of electrons and of protons respectivel Due to its exponential dependence in Eq. (10), the most important effect on the ionization fluctuation come from fluctuations in the temperature. From Eq. (10) it follows

$$\delta x = [x(1-x)/4(2-x)]\,[(3/2) + (\chi/k_B\,T_o)]\,(\delta\rho_r/\rho_{ro}) \; . \qquad (11)$$

ı Figure 1, $x$ is plotted as a function of $z$ for different values of $\Omega h^2$.

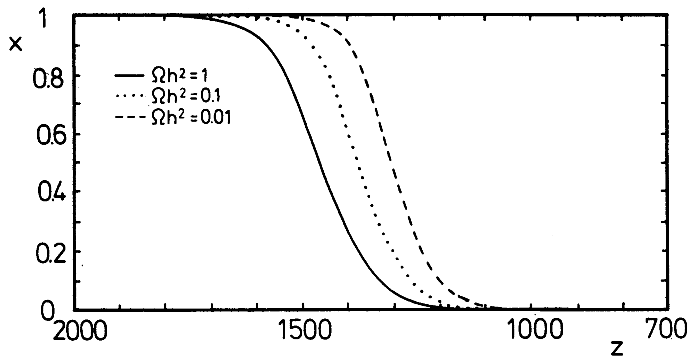


Fig. 1. Ionization degree as function of $z$ for different values of $\Omega h^2$.

Substituting Eq. (11) into Eq. (9) yields

$$\delta p_g = (1+x)(k_B\,T_o/m_H)\,\delta\rho_g + \{\,(1+x)/4 + [x(1-x)/4(2-x)]\,[(3/2) + (\chi/k_B\,T_o)]\}\,(\rho_{go}\,k_B\,T_o/m_H)\;(\delta\rho_o/\rho_{ro}) \; . \qquad (12)$$

ıserting Eq. (12) in Eq. (3), the density contrasts

$$\delta_g = \delta\rho_g/\rho_{go} \; ,$$

$$\delta_r = \delta\rho_r/\rho_{ro} \; ,$$

ıd the time scale

$$dt = (4\pi G\,\tilde\rho)^{-1/2}\,d\tau \; ,$$

here $\tilde\rho$ is the matter density at $z = 2000$; the following system of equations can be obtained

$$d\delta\,\tilde v^i_g/d\tau + \left[2\,H + (4\,R/3\,\tau_o)\right]\left[\delta\tilde v^i_g/(4\pi G\,\tilde\rho)^{1/2}\right] - \left[4\,R\delta\tilde v^i_r/(3\,\tau_o(4\,\pi G\,\tilde\rho)^{1/2})\right] + + \left[p_{go}\,\delta_{g,i}/(a^2\,(4\pi G\,\tilde\rho)^{1/2})\right] + \left\{1/4 + \left[x(1-x)/4(2-x)(1+x)\right][\,3/2 + (\chi/k_B\,T_o)\,] \times \times\left[p_{go}/a^2\,(4\pi G\,\tilde\rho)\right]\delta_{r,i} = 0 , \qquad (13)$$

$$d\delta\tilde v^i_r/d\tau + [H + (1/\tau_0)]\left[\delta\tilde v^i_r/(4\pi G\,\tilde\rho)^{1/2}\right] - - \left[\delta\tilde v^i_g/(\tau_o(4\pi G\,\tilde\rho)^{1/2})\right] - (3/4)\left[c_s^2/(a^2(4\pi G\,\tilde\rho))\right]\delta_{r,i} = 0 \; , \qquad (14)$$

$$d\delta_g/d\tau + 3\,H\,\delta_g(4\pi G\,\tilde\rho)^{-1/2} + \delta\tilde v^i_{;i} + (1/2a^2)\,(d\,h_{ii}/d\tau) - H\,(h_{ii}/a^2)(4\pi G\,\tilde\rho)^{-1/2} = 0 \; , \qquad (15)$$

$$d\,\delta_r/d\tau + 4\,H\,\delta_r(4\pi G\,\tilde\rho)^{-1/2} + (4/3)\,\delta v^i_{r,i} + (2/3\,a^2)\,(d\,h_{ii}/d\tau) = 0 \; , \qquad (16)$$

$$(d^2\,h_{ii}/d\tau^2) + 2(4\pi G\,\tilde\rho)^{-1/2}\,H\,(d\,h_{ii}/d\tau) = -2(\rho_{go}/\tilde\rho)(\delta_g + 2\,R\,\delta_r)\, ; \qquad (17)$$

where $\rho_{go}$ represents the unperturbed pressure, $\delta\tilde{v}_g^i = \delta v_g^i(4\pi G\tilde{\rho})^{-1/2}$ and $\delta\tilde{v}_r^i = \delta v_r^i(4\pi G\tilde{\rho})^{-1/}$ This system of equations characterizes our model of the Universe. Before solving it, we will discuss th calculation of the expansion parameter $a(t)$.

## III. INTEGRATION OF FRIEDMANN'S COSMOLOGICAL EQUATION

The cosmic scale factor $a(t)$ of the unperturbed metric for a plane Universe obeys the Friedmann equatio

$$(\dot{a}/a)^2 = 8\pi G\rho/3, \qquad (18)$$

where $\rho$ includes the matter and radiation density. By substituting

$$\rho_g = \rho_{go}\,(a_o/a)^3\ ,\ \rho_r = \rho_{ro}\,(a_o/a)^4$$

into Eq. (17), we obtain

$$(\dot{a}/a)^3 = (8\pi G/3)\,[\rho_{go}\,(a_o^3/a^3) + \rho_{ro}\,(a_o^4/a^4)]\ .$$

We change now the variable $t$ through

$$c\,dt = a\,d\eta\ , \qquad (19)$$

and get

$$(da/d\eta)^2 = (8\pi G/3\,c^2)\,(\rho_{go}\,a_o^3\,a + \rho_{ro}\,a_o^4)\ . \qquad (20)$$

Integration of Eq. (20) yields

$$a(\eta) = (8\pi G/12\,c^2)\,\rho_{go}\,a_o^3\,\eta^2 + [(8\pi G/3\,c^2)\,\rho_{go}\,a_o^4]^{1/2}\,\eta\ ; \qquad (21)$$

and with the aid of expression (21) we can integrate Eq. (19) and obtain

$$t = (8\pi G/36\,c^3)\,\rho_{go}\,a_o^3\,\eta^3 + (1/2\,c)\,[\,(8\pi G/3\,c^2)\,\rho_{ro}\,a_o^4\,]^{1/2}\,\eta^2\ . \qquad (22)$$

## IV. SOLUTION OF THE EQUATIONS OF MOTION

A qualitative analysis of equations (13)-(17) in order to know if they provide solutions with physical sense can be carried out as follows: for a static Universe $a = 1$ one obtains the following two coupled wave equation of the matter and radiation density perturbations

$$\delta\,\ddot{\rho}_m + (4/3)\,(R/\tau_o)\delta\,\dot{\rho}_m + (k^2\,v_s^2 - 4\pi G\,\rho_{om})\,\delta\rho_m - \Big[(\delta\dot{\rho}_r/\tau_o) + 8\pi G\rho_{om}\,\delta\rho_r\Big] = 0\ , \qquad (23)$$

$$\delta\ddot{\rho}_r + (\delta\dot{\rho}_r/\tau_o) + [k^2\,c_s^2 - (32/3)\,\pi G\,\rho_{om}\,R]\delta\,\rho_r - (4/3)\,R[(\delta\dot{\rho}_m/\tau_o) + 4\pi G\,\rho_{om}\,\delta\rho_m] = 0\ , \qquad (24)$$

where $\tau_o$ is the collision time, $v_s$ is the sound velocity of the matter fluid and $c_s = c/(3)^{1/2}$ is the velocity c sound for a pure photon gas. With the *ansatz* $\mathbf{A} \sim exp(\omega t + i\vec{k}\,\vec{r})$, where $\mathbf{A}$ is a vectorial abbreviation for th unknowns $\delta\rho_m$ and $\delta\rho_r$ we obtain the dispersion relation

$$\begin{aligned}&\omega^4 + \omega^3\,[1 + (4/3)\,R]/\tau_o + \omega^2\,\{\,k^2\,(c_s^2 + v_s^2) - 4\pi G\,\rho_{om}\,[1 + (8/3)\,R]\,\} + \\ &\omega\,\Big(k^2\,v_s^2 - 4\pi G\,\rho_{om}\,[1 + (8/3)\,R] + (4/3)\,R\,\Big\{k^2\,c_s^2 - 4\pi G\,\rho_{om}\,[1 + (8/3)\,R]\Big\}\Big)/\tau_o + \\ &+ k^2\,v_s^2\,[k^2\,c_s^2 - (32/3)\,\pi G\,\rho_{or}] - 4\pi G\,\rho_{om}\,k^2\,c_s^2 = 0\ . \end{aligned} \qquad (25)$$

Applying the Routh-Hurwitz criterion (Corona-Galindo 1985) for finding the instability conditions, we obtain the wave numbers

$$k_1^2 = [4\pi G/(c_s^2 + v_s^2)]\, \rho_{om}\, [1 + (8/3)\, R] \;, \tag{26}$$

and

$$k_2^2 = b/a \;, \tag{27}$$

where

$$b = (4\pi G/\tau_o^2)\, \rho_{om} \Big( [1 + (8/3)\, R][1 + (4/3)\, R] \{[(c_s^2 + v_s^2)/v_s^2]\, [(1 + (4/3)\, R] - - (4R\, c_s^2\, /3\, v_s^2)\, [1 + (8/3)\, R] - 1\} + (c_s^2/v_s^2)\, [1 + (4/3)\, R]^2 \Big) \;; \tag{28}$$

$$a = (v_s^2/\tau_o^2)[1 + (4/3)\, R\, (c_s^2/v_s^2)] \{ [(c_s^2 + v_s^2)/v_s^2]\, [1 + (4/3)\, R] - - (4\, R\, c_s^2/3\, v_s^2)\, [1 + (8/3)\, R] - 1\} . \tag{29}$$

By comparing $k_1$ and $k_2$ we obtain immediately the condition given by $k_1$ as the Jeans instability condition. It means that the equations proposed here lead to acceptable solutions from the cosmological point of view.

To solve equations (13)-(17) dynamically we assume the solutions are of the form

$$\begin{aligned} \delta_g &= A(\tau)\, exp(i\, k_a\, x^a) \;, \\ \delta_r &= T(\tau)\, exp(i\, k_a\, x^a) \;, \\ \delta \tilde{v}_g^j &= V^j(\tau)\, exp(i\, k_a\, x^a) \;, \\ \delta\, \tilde{v}_r^j &= W^j\, (\tau)\, exp(i\, k_a\, x^a) \;, \\ h_{ii} &= H(\tau)\, exp(i\, k_a\, x^a) \;, \\ d\, h_{ii}/d\, \tau &= G_{ii} = G(\tau)\, exp(i\, k_a\, x^a) \;. \end{aligned} \tag{30}$$

Substituting the above solutions into (13)-(17) and assuming the velocity perturbations to be irrotational, i.e., derivable from a potential $v \parallel k$, the following six linear differential equations for the unknows $A(\tau)$, $T(\tau)$, $\varsigma(\tau) = ik_jV^j(\tau)$, $\xi(\tau) = ik_jW^j(\tau)$, $H(\tau)$ and $G(\tau)$:

$$d\varsigma(\tau)/d\tau + [3\, H + (4\, R/3\, \tau_o)]\, [\varsigma(\tau)/\alpha] - (4\, R/3\, \tau_o\, \alpha)\, \xi(\tau) - \rho_{go}\, [k^2\, A(\tau)/(a^2\, \alpha^2\, \rho_{go})] - - \{ 1/4 + [x(1-x)/4(2-x)(1-x)]\, [3/2 + (\chi/k_B\, T_o)]\}\, (p_{go}\, k^2/(a^2\, \alpha^2\, p_{go}))\, T(\tau) = 0 \;, \tag{31}$$

$$d\xi(\tau)/d\tau + [2H + (1/\tau_o)]\, [\xi(\tau)/\alpha] - (1/\tau_o\alpha)\, \varsigma(\tau) - (3\, c_s^2\, k^2/4\, a^2\, \alpha^2)\, T(\tau) = 0 \;, \tag{32}$$

$$dA(\tau)/d\tau + (3H/\alpha)\, A(\tau) + \xi(\tau) + (1/2a^2)\, G(\tau) - (H/a^2\, \alpha)\, H(\tau) = 0 \;, \tag{33}$$

$$dT(\tau)/d\tau + (4H/\alpha)\, T(\tau) + (4/3)\, \xi(\tau) + (2/\, 3a^2)\, G(\tau) = 0 \;, \tag{34}$$

$$dG(\tau)/d\tau + (2H/\alpha)\, G(\tau) + (8\pi G/\alpha^2)\, \rho_{go} \Big[ A(\tau) + 2R\, T(\tau) \Big] = 0 \;, \tag{35}$$

and

$$dH(\tau)/d\tau = G(\tau) ; \qquad (36)$$

where we have written $\alpha = (4\,\pi G\rho)^{1/2}$, and $R = \rho_{ro}/\rho_{go}$.

## V. DISCUSSION OF THE RESULTS

The system of equations (31)-(36) was solved numerically and the solutions are plotted in Figures 2 and 3 for different masses. In Figures 2 and 3 one can appreciate the decreasing behavior of the velocities $\tilde{\xi}(\tau)$, $\tilde{\zeta}(\tau)$, the perturbation of gravitational potential $\tilde{H}(\tau)$ and its derivative $\tilde{G}(\tau)$, the matter density $A(\tau)$, and radiation density-fluctuation $T(\tau)$, of the mixture matter-radiation by increasing time $\tau$. For small masses ($10^7$ $M_\odot$, $10^9$ $M_\odot$) the velocities and density contrasts decrease almost exponentially, but for larger masses ($10^{13}$ $M_\odot$, $10^{16}$ $M_\odot$) the radiation quantities change very smoothly. The initial values used in order to obtain these figures

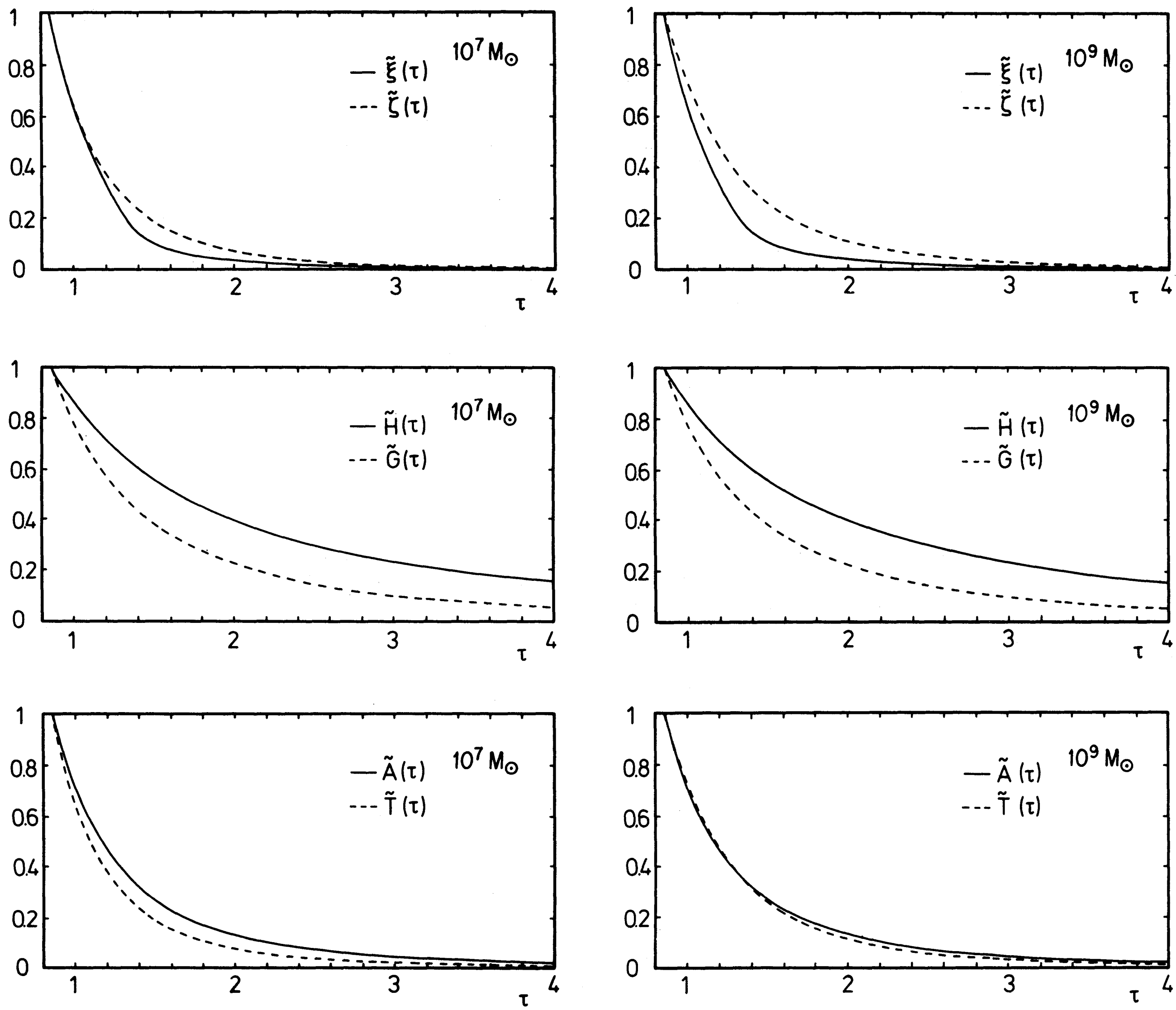


Fig. 2. Time evolution of velocity, metric perturbations and density contrasts for $10^7$ $M_\odot$ and $10^9$ $M_\odot$.

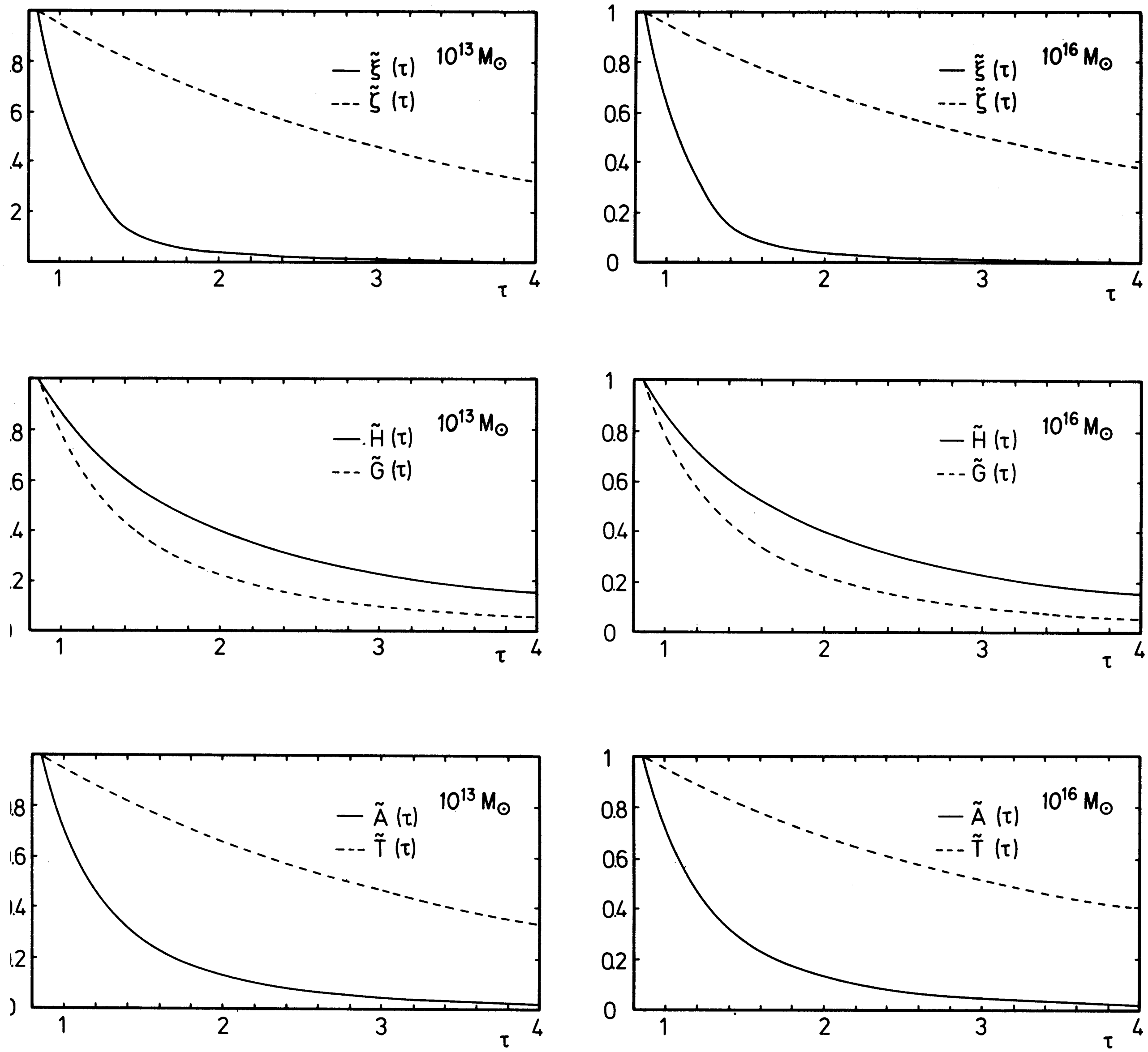


Fig. 3. The same as in Figure 1, but for $10^{13}$ and $10^{16}$ solar masses.

ere: $\xi_o(0.8) = 10^{-6}$, $\varsigma_o(0.8) = 10^{-5}$, $H_o(0.8) = 10^{-13}$, $G_o(0.8) = 10^{-13}$, $A_o(0.8) = 10^{-6}$, and $T_o(0.8) =$ )$^{-12}$. More values were used, but only decreasing behaviors were found. Although the results are not very ɔod from the cosmological point of view, they show however the validity of the method of integration of the ynamic equations of motion which govern fluids in expansion. The method of integration of the equations resented here, has applicability in all the cosmological models which take into account the expansion of the niverse or in hydrodynamic models governed by equations of motion with non-constant coefficients, e.g. ıe hydrodynamic treatment of the terrestrial atmosphere.

I am very grateful to Miss Arroyo-Castelazo for typing the manuscript.

## REFERENCES


Corona, M.G. 1987, RevMexAA, 14, 52

Corona, M. 1985, in Hydrodynamical Approach of Galaxy Formation on the Basis of a Fluid-Model of two and three Components during the Recombination Era of the Universe, (Konstanz:Hartung-Gorre Press), p. 122

Nowotny, E. 1981, Ap&SS, 76, 235

Lifshitz, E.M. 1946, J. Phys. USSR, 10, 116

Lifshitz, E.M., & Khalatnikov, I.M. 1963, Adv. Phys., 12, 185

Peebles, P.J.E. 1968, ApJ, 153, 1

Rose, B., & Corona-Galindo, M.G. 1991, Gen. Rel. Grav., 23, 1317

Weinberg, S. 1972, in Gravitation and Cosmology (New York:J. Willey and Sons)


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