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# ON THE ANALYTICAL BEHAVIOR OF THE WHITROW–RANDALL RELATION IN BRANS–DICKE GRAVITY

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RESUMEN

Algunos modelos cosmológicos en la teoría de Brans-Dicke (BD) de gravitación, que satisfacen la relación de Whitrow-Randall (WRR) como condición ad hoc, han sido recientemente analizados. Aquí presentamos un estudio análitico del comportamiento de esta relación, usando las soluciones cosmológicas más simples de la teoría. Se muestra que la relación WRR es válida para un amplio rango de valores del parámetro $w$.

ABSTRACT

Brans Dicke cosmological models satisfying the Whitrow Randall Relation (WRR) ad hoc have recently been discussed. We present here an analytical study of the behavior of the Whitrow-Randall Relation (WRR) in Brans-Dicke gravity. We use the simplest cosmological solutions to show that the WRR relation holds for a large interval of values of the $w$ parameter.

*Key words*: **COSMOLOGY – THEORY — GRAVITATION**

## 1. INTRODUCTION

The geometrical and inertial properties of space have been a subject of discussion for quite a long time. There are basically two major different approaches. In Newton's picture, space has properties of its own, whereby the introduction of the concept of an absolute reference system follows naturally. On the other hand, in Mach's view (Mach 1883) space is solely justified by its matter content. Both pictures are very old and can be traced back to the writings of Berkeley (1710). According to Mach, the inertial properties of a body may be interpreted as a gravitational effect due to the action of distant matter. This statement is known as Mach's Principle. Einstein was seduced by the ideas of Mach, and in fact, General Relativity (GR) was constructed to be consistent with Mach's Principle (Einstein 1951). However, he did not succeed completely, and one of the most severe problems is the fact that, although in GR the geometry of space is influenced by the distribution of mass, it is not uniquely determined by it; therefore, the vacuum solutions cannot be eliminated. In the decades after the birth of GR, other theories of gravity were built in order to conciliate Mach's ideas on the notion of space (Whitrow & Randall 1951; Sciama 1953). The most serious attempt of constructing such a theory was carried out by Brans & Dicke (1961, BD). They developed a relativistic theory of gravitation based on the existence of a scalar field in a Riemannian geometry. The gravitational constant was then reborn as a field variable.

A relation between the mass, the radius of the visible universe and the value of the *gravitational constant* $G$ seems to hold whenever the Mach's Principle is valid (Whitrow & Randall 1951). That is

$$\frac{GM}{R} \sim 1 \ . \qquad (1)$$

This relation is known as Whitrow-Randall Relation (WRR), and can also be obtained by dimensional analysis under the assumptions of the Mach's Principle (Dicke 1959).

In this report we will study the evolution in time of the WRR in the framework of BD theory of gravity. Mimoso (1993) has shown that the solutions of a general scalar-tensor theory may be reduced to that of BD using the Mean Value Theorem of Integral Calculus and an adequate definition of the coupling constant $w$. Therefore, this enhances our interest in the analysis of the problem in the particular case of BD gravity. So far, the *analytical study* of the WRR in a BD theory has not been carried out. Instead, only solutions which satisfy this relation exactly have been introduced up to date. These solutions were presented by Berman & Som (1990) and corrected by Beesham (1995). They both show that if the WRR were satisfied exactly, power law type cosmological solutions would arise. This last result suggests that we adopt the opposite point of view; that is, not to ask for an exact fulfillment of the WRR but instead, study if it is automatically satisfied by the usual and well known BD cosmological solutions.

The paper is organized as follows. In section § 2, the formal equations of the BD theory are introduced. Section 3 presents the calculation of the horizon size and the mass inside the horizon in a two-epochs universe scenario, and finally in § 4, we show and analyze the behavior of the WRR.

## 2. SCALAR-TENSOR THEORIES OF GRAVITY (STTG)

STTG can be derived from the action

$$S = \frac{1}{16\pi}\int \sqrt{-g}dx^4 \left[\phi R - \frac{w(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right] + S_M \,, \tag{2}$$

where $R$ is the Ricci curvature scalar, $\phi$ is a scalar field, $w(\phi)$ is a dimensionless function of $\phi$ which gives the strength of the coupling between the scalar field and the metric, and $S_M$ stands for the matter action. BD gravity appears as a special case when the coupling function $w(\phi)$ is a constant.

Applying the least action principle in a Friedmann-Robertson-Walker ($k = 0$) universe, and using as the equation of state that of a perfect fluid, the BD field equations become

$$\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a}{a}\frac{\dot\phi}{\phi} = \frac{w}{6}\left(\frac{\dot\phi}{\phi}\right)^2 + \frac{8\pi}{3}\frac{1}{\phi}\rho \,, \tag{3}$$

$$2\frac{d}{dt}\left(\frac{\dot a}{a}\right) + 3\left(\frac{\dot a}{a}\right)^2 + 2\frac{\dot a}{a}\frac{\dot\phi}{\phi} = -\frac{w}{2}\left(\frac{\dot\phi}{\phi}\right)^2 - \frac{8\pi}{\phi}p - \frac{\ddot\phi}{\phi} \,, \tag{4}$$

$$\ddot\phi + 3\frac{\dot a}{a}\dot\phi = -\frac{1}{2w+3}8\pi(3p-\rho) \,, \tag{5}$$

while the divergence of the stress-energy tensor of matter yields

$$\dot\rho + 3\frac{\dot a}{a}(p+\rho) = 0. \tag{6}$$

Note that for the radiation era, $\phi = constant$ is always a solution of (5) because this equation becomes sourceless. Although this is only a particular solution (Gurevich, Finkelstein, & Ruban 1973; Barrow 1993), it will be sufficient to take this one for our purposes. This will also allow more clarity in the computations that follow.

## 3. HORIZON SIZE AND MASS INSIDE THE HORIZON

We consider the following 2-epochs cosmological solution of the flat FRW Brans-Dicke model

$$a^{[1]}(t) = A_1 t^{1/2}, \qquad 0 < t \le t_{eq}, \tag{7}$$

and

$$a^{[2]}(2) = A_2 (t - t_2)^{\frac{2w+2}{3w+4}} \,, \qquad t_{eq} \le t \le t_0. \tag{8}$$

The first represents a solution for the radiation dominated era (between the Planck time and the equivalence time $t_{eq}$) while the second corresponds to the matter dominated era, first developed by Nariai (1968). This two-epochs cosmology, with the solutions quoted above, has been used in many computations within BD gravitation, in particular for nucleosynthesis studies see (Casas, García-Bellido, & Quirós 1992; Torres 1995). Imposing a smooth matching of the solutions at $t = t_{eq}$, we obtain

$$A_2 = A_1\left[\frac{3w+4}{4w+4}\right]^{\frac{2w+2}{3w+4}} t_{eq}^{-\frac{w}{2(3w+4)}}, \tag{9}$$

$$t_2 = -\frac{w}{3w+4}t_{eq}. \tag{10}$$

The solutions for the scalar field are

$$\phi^{[1]}(t) = \phi_0, \qquad 0 < t \le t_{eq}, \tag{11}$$

$$\phi^{[2]}(t) = \phi_1 (t - t_2)^{\frac{2}{3w+4}} \,, \qquad t_{eq} \le t \le t_0, \tag{12}$$

where $t_0$ stands for the time today, and $\phi_0$ and $\phi_1$ are dimensional constants.

Note that the time of equivalence is defined as that of GR. The usual definition of this time is given by the equality of the energy densities of matter and radiation. However, in order to obtain a value for this time it is necessary to use the complete solution of the radiation plus matter model. This solution is only known for General Relativity. In fact, in (Torres & Helmi 1996) we have shown that it is not possible to define such a time in a large set of BD theories of gravity and probably neither in the rest of the scalar tensor theories. It means that in this context, the time selected acts only as a parameter of the theory with no other physical meaning than the smooth matching of the solutions. This will be represented by a discontinuity in the WRR, given by a discontinuity in the energy densities between the two eras.

The particle horizon size is defined as

$$d_H = a(t) \int_0^t \frac{d\tau}{a(\tau)}. \qquad (13)$$

So, we calculate (13) using (7) and (8) for each of the two epochs, respectively, obtaining finally

$$d_H^{[1]} = 2t, \qquad (14)$$

$$d_H^{[2]} = \frac{3w+4}{w+2} t_{eq} \left[ \frac{t}{t_{eq}} + \frac{w}{3w+4} \right] - \left( \frac{3w+4}{4w+4} \right)^{\frac{2w+2}{3w+4}} \frac{2w}{w+2} t_{eq} \left[ \frac{t}{t_{eq}} + \frac{w}{3w+4} \right]^{\frac{2w+2}{3w+4}} \qquad (15)$$

The mass inside the horizon is computed as

$$M = \frac{4}{3} \pi \, d_H^3 \, \rho, \qquad (16)$$

where the density $\rho$ is derived from the field equations (3), giving

$$\rho^{[1]}(t) = \frac{3\phi_0}{8\pi} \frac{1}{4t^2}, \qquad (17)$$

$$\rho^{[2]}(t) = \frac{3\phi(t)}{8\pi} \frac{1}{(t-t_2)^2} \times \left[ \left( \frac{2w+2}{3w+4} \right)^2 + \frac{4w+4}{(3w+4)^2} - \frac{w}{6} \left( \frac{2}{3w+4} \right)^2 \right]. \qquad (18)$$

## 4. RESULTS AND CONCLUSIONS

The analytical behavior of the WRR is obtained using (14) and (15) together with (17) and (18), and replacing them into equation (1). Figure 1 represents the relation for different values of $w$ as a function of $t$. Figure 2 shows the value of the WRR today as a function of the coupling constant.

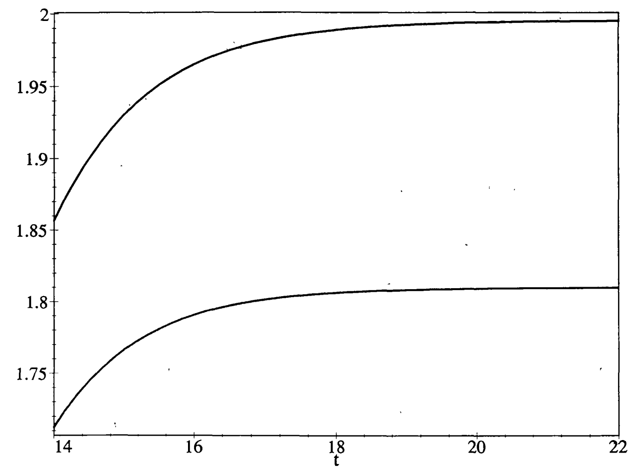


Fig. 1. Whitrow Randall relation as a function of time in the matter dominated era of Brans Dicke cosmology, shown for two different values of the coupling constant. The upper curve corresponds to $w = 500$ and the lower one to $w = 10$. The axis $t$ stands for the base logarithm of the cosmological time.

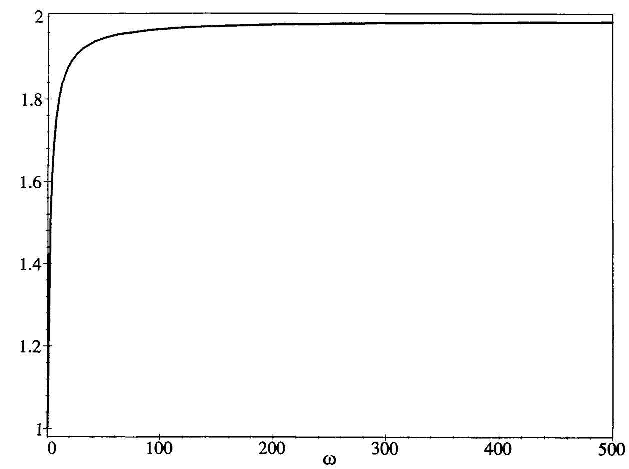


Fig. 2. Whitrow Randall relation value for today, as a function of the Brans Dicke coupling constant $w$.

This simple exercise shows that the WRR holds true when the simplest cosmological solutions of the BD theory are taken into account. Other similar estimates can be obtained by means of different solutions; for instance, for the radiation era. Anyway, in previous works (Berman & Som 1990; Beesham 1995), the study of the BD cosmology was done by imposing a constant deceleration parameter and the WRR to hold exactly. In this way, analytical solutions were found. Although these solutions resulted to be of power law type, in some cases the coupling constant $w$ had to take values which are not compatible with either bounds imposed by the inflationary scenario or from primordial nucleosynthesis. It is, therefore, shown that the usual and simplest BD solutions satisfy the WRR for a large interval of values of $w$, without the need of introducing ad-hoc ansatzs.

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