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THE PARAMETERS OF GALACTIC KINEMATICS DETERMINED FROM TOTAL LEAST SQUARES

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RESUMEN

Los problemas de la cinemática Galáctica generalmente se han resuelto usando el método de mínimos cuadrados, pero éste puede calcular resultados sesgados porque mínimos cuadrados asumen que hay error únicamente en las observaciones, no en las ecuaciones de condición. Pero éstas, sin embargo, a menudo incorporan error, por lo menos en algunas de las columnas de la matriz de las ecuaciones. Mínimos cuadrados totales representan una herramienta matemática idónea para justamente esta clase de problema. En este trabajo se emplea el método, o mejor dicho un mezcla de mínimos cuadrados ordinarios-totales, con mas de 100,000 estrellas tomadas del catalógo *Hipparcos* para calcular doce parámetros de la cinemática Galáctica: nueve componentes del tensor de deformación, relacionadas con entidades como las constantes A y B de Oort, el término K, el tensor de rotación Galáctico y las componentes del movimiento solar. Se obtienen valores razonables para todas las incógnitas, demostrando así que datos astrométricos de alta calidad y un método adecuado de reducción de los datos producen buenos resultados cuando en el pasado esto ha sido difícil o imposible.

ABSTRACT

Problems of Galactic kinematics have usually been solved by the method of least squares, but this may lead to biased results because least squares assume that error resides only in the observations, not in the equations of condition. The latter, however, generally incorporate error, at least in some of the columns of the data matrix. Total least squares represents the ideal mathematical tool for just this sort of problem. In this paper, the method, or better stated a mixed total-ordinary least squares method, is applied to over 100,000 stars taken from the *Hipparcos* catalog to calculate twelve parameters of Galactic kinematics: the nine components of the deformation tensor, related to such quantities as the Oort A and B parameters, the K term, Galactic vorticity, and the Solar motion. Reasonable values for all of these is obtained, showing that high quality astrometric data and an adequate reduction method can produce good results for global solutions when in the past this proved difficult or impossible.

Key Words: **GALAXY: FUNDAMENTAL PARAMETERS — GALAXY: KINEMATICS AND DYNAMICS — METHODS: DATA ANALYSIS**

1. INTRODUCTION

In a previous publication, I have shown how the mathematical technique known as total least squares (TLS) is an ideal tool for work with Galactic kinematics (Branham 1998). That publication, however, based on stars taken from the Yale Bright Star catalog (BSC5), merely shows the power of TLS and was not intended as a contribution to the determination of the parameters of Galactic kinematics. Now that the *Hipparcos* catalog (ESA 1997) has been published, to use the new parallaxes for a genuine determination of these parameters seems indicated. TLS also seems indicated as the relevant reduction technique. I will not explain the fundamentals of the method because these have been detailed in the previous publication and also in others (Branham 1995, 1997) but will merely summarize the salient features.

We have a linear system with m equations of condition and n unknowns. Let A be the matrix of the equations of condition (also called data matrix), of size $m \times n$, x an n-vector of the solution, and d an m-vector of the observations

$$A \cdot x = d. \tag{1}$$

If error exists only in d the problem is one of ordinary least squares (OLS). But if A also contains error, the problem is one of TLS. Because of this error equation (1) is inconsistent: no single vector x will satisfy all of the equations of condition. TLS proposes as the solution the vector x that minimizes the Frobenius norm of the error in A and d. If we denote this error by ΔA and Δd then we seek

$$\| \Delta A : \Delta d \|_F = \; min, \tag{2}$$

where F denotes the Frobenius norm and the colon indicates that Δd is appended as an additional column to ΔA. Golub & Van Loan (1980) show that the minimization implied by equation (2) is achieved by calculating a vector x that solves

$$(A^T \cdot A - \sigma^2 I)x = A^T \cdot d, \tag{3}$$

where I is the unit matrix and σ^2 is the smallest eigenvalue of the matrix $(A : d)^T \cdot (A^T : d)$. Golub & Van Loan (1980) actually obtain the solution by us of the singular value decomposition (SVD) rather than eigenvalues, but unless A is poorly conditioned, equation (3) calculates the solution with less memory and fewer operations (Branham 1989).

If some of the columns of A are error-free, the problem becomes one of mixed TLS and least squares, TLS-LS. For a discussion of the theory behind TLS-LS, see Van Huffel & Vandewalle (1991). For a brief description of the technique to employ let k of the columns of A be error-free. Permute the columns of A so that its first k columns are these error-free columns. Then the TLS-LS solution is given by

$$x = (A^T \cdot A - \sigma^2 \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix})^{-1} A^T \cdot d, \tag{4}$$

where I_{n-k} refers to the $(n-k) \times (n-k)$ identity matrix. What about σ^2? Form the matrix $B = [A : d]^T \cdot [A : d]$ and partition it so that

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where B_{11}, of size $k \times k$, corresponds to error-free columns of $[A : d]$, B_{21}, of size $k \times (n-k+1)$, and B_{12}, of size $(n-k+1) \times k$ and equal to B_{21}^T, a mixture of error-free and error columns, and B_{22}, of size $(n-k+1) \times (n-k+1)$, to pure error columns. σ^2 is the smallest eigenvalue of $B_{22} - B_{21} \cdot B_{11}^{-1} \cdot B_{12}$, the Schur complement of B_{11} in B.

Equations (3) and (4) show that the solution changes when σ changes. Because σ varies according to the scaling of the data matrix, a TLS or mixed TLS-LS solution, unlike an OLS solution, depends on how the data matrix is scaled. I have found that column scaling or both row and column scaling, but seldom row scaling alone, of A works well. No scaling at all performs poorly.

However, the solution is computed, OLS, TLS, or mixed TLS-LS, the covariance matrix, by permitting the calculation of mean errors and correlations, constitutes vital additional information to the solution. See Branham (1999) for a method for calculating the covariance matrix.

2. TLS AND GALACTIC KINEMATICS

Galactic kinematics represents one area where mixed TLS-LS should be ideal. To see why, let us look at typical equations of condition. Use the Ogorodnikov-Milne model, truncated after the first order in Taylor series expansions for the velocity of the centroid of a group of stars, which is valid for distances up to about 1 kpc (Ogorodnikov 1965). Distances will be measured in parsecs, proper motions in milli-arc-sec (mas) per year, and radial velocities in kilometers per second. Let R_0 be the centroid's distance to the Galactic center, r the distance from the center of the centroid (the Sun) to a star, V the velocity of the centroid at distance R, and V_0, the reflex solar motion, its velocity at distance R_0. Then from elementary calculus we have

$$V = V_0 + D \cdot r, \tag{5}$$

where D is the displacement tensor of partial derivatives evaluated at R_0

$$D = \begin{pmatrix} \partial V_x/\partial x & \partial V_x/\partial y & \partial V_x/\partial z \\ \partial V_y/\partial x & \partial V_y/\partial y & \partial V_y/\partial z \\ \partial V_z/\partial x & \partial V_z/\partial y & \partial V_z/\partial z \end{pmatrix}_{R=R_0} = \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}. \tag{6}$$

Equation (5) involves a total of twelve unknowns, the three components of the reflex solar motion and the nine components of the displacement tensor.

That the Ogorodnikov-Milne model should be adequate becomes manifest upon looking at the distribution of the parallaxes from the *Hipparcos* catalog. Of the 118,310 parallaxes, 113,710 are nonnegative. The negative parallaxes were excluded in this study. Of the positive parallaxes, only 6300 (5.6%) are smaller than 1 milli-arc-second, and these have a median error of 1.11 = 20 mas. Because 1 mas corresponds with a distance of 1 kpc, the vast majority of the stars are at distances that comply with the assumptions of the Ogorodnikov-Milne model, and the remainder have such large errors that it becomes difficult to say what their true distances are. This, however, is a situation where TLS, by taking account of the error in the equations of condition, seems indicated as the reduction technique.

The displacement tensor D may be decomposed into the sum of a symmetric tensor S, the strain tensor,

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(w_y + v_z) \\ \frac{1}{2}(u_z + w_x) & \frac{1}{2}(w_y + v_z) & w_z \end{pmatrix}, \tag{7}$$

and an antisymmetric tensor Ω, the rotation tensor,

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2}(u_y - v_x) & \frac{1}{2}(u_z - w_x) \\ -\frac{1}{2}(u_y - v_x) & 0 & \frac{1}{2}(v_z - w_y) \\ -\frac{1}{2}(u_z - w_x) & -\frac{1}{2}(v_z - w_y) & 0 \end{pmatrix}. \tag{8}$$

From these two tensors various quantities usually associated with Galactic kinematics, such as the Oort A and B constants, may be derived once certain assumptions are made.

To convert equations (5) to (8) into a form more suitable for computation, use Galactic coordinates (l, b) for positions, set the distance $r = 1/\pi$, where π is the trigonometric parallax from the *Hipparcos* catalog, and define the direction cosines

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} \cos l \cos b \\ \sin l \cos b \\ \sin b \end{pmatrix}; \begin{pmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{pmatrix} = \begin{pmatrix} -\sin l \\ \cos l \\ 0 \end{pmatrix}; \begin{pmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} -\cos l \sin b \\ -\sin l \sin b \\ \cos b \end{pmatrix}. \tag{9}$$

The equations may be expressed, if (X, Y, Z) denotes the components of the reflex solar velocity, as equations of condition in radial velocity, $\dot{r}$, proper motion in Galactic longitude, μ_l, and proper motion in Galactic latitude, μ_b

$$\dot{r} = \frac{1}{\pi}(\alpha^2 u_x + \alpha\beta u_y + \alpha\gamma u_z + \alpha\beta v_x + \beta^2 v_y + \beta\gamma v_z + \alpha\gamma w_x + \beta\gamma w_y + \gamma^2 w_z) \\ -\alpha X - \beta Y - \gamma Z; \tag{10}$$

$$\kappa\mu_l = \frac{\sec b}{\pi}(-\alpha\beta u_x - \beta^2 u_y - \beta\gamma u_z + \alpha^2 v_x + \alpha\beta v_y + \alpha\gamma v_z) + \alpha_1 X + \beta_1 Y + \gamma_1 Z; \tag{11}$$

$$\kappa\mu_b = \frac{-\sec b}{\pi}[\alpha^2\gamma u_x + \alpha\beta\gamma u_y + \alpha\gamma^2 u_z + \alpha\beta\gamma v_x + \beta^2\gamma v_y + \beta\gamma^2 v_z + \\ \alpha(\gamma^2 - 1)w_x + \beta(\gamma^2 - 1)w_y - \gamma(\alpha^2 + \beta^2)w_z] + \alpha_2 X + \beta_2 Y + \gamma_2 Z, \tag{12}$$

where κ represents a conversion constant with value 4.74047 km s^{-1} yr. Notice that not all of the unknowns in equations (10) to (12) are independent: in equation (10) only nine of the unknowns are independent; in equation (12) eleven are independent, and equation (11) includes none of the unknowns w_x, w_y, or w_z. Thus, if one were to use the equations separately, it would not be possible to solve for all of the unknowns in a given equation. But this stricture no longer applies if the equations are used together, when all of the unknowns become independent.

The presence of the parallax π on *the right-hand-side*, not merely in the data on the left-hand-side, constitutes an important consideration for solving in equations (10) to (12), which therefore, represents a situation where the assumptions of OLS become invalid. The errors of parallax measurements are substantial, typically 23% of the value itself. (This percentage comes from the median parallax and median parallax error in the *Hipparcos* catalog). Thus, in the equations of condition the coefficients of $u_x - w_z$ incorporate significant error whereas the coefficients of X, Y, Z are basically error-free. To solve equations (10) to (12) one should use mixed TLS-LS rather than OLS or even pure TLS. But before the equations can be solved by mixed TLS-LS they should be recast to avoid a Lutz-Kelker effect. Smith & Eichhorn (1996), moreover, have demonstrated that the distances to stars have an error distribution that is skewed and the direction of whose bias depends on the size of the standard deviation relative to the true value. To avoid these difficulties one should, therefore, multiply eqs. (10) to (12) by π to transfer the error from the denominator. Then the error-free unknowns become $u_x - w_z$ and the error unknowns X, Y, Z. The transformed equations are

$$\pi\dot{r} = (\alpha^2 u_x + \alpha\beta u_y + \alpha\gamma u_z + \alpha\beta v_x + \beta^2 v_y + \beta\gamma v_z + \alpha\gamma w_x + \beta\gamma w_y + \gamma^2 w_z) \\ -\pi\alpha X - \pi\beta Y - \pi\gamma Z; \tag{13}$$

$$\kappa\pi\mu_l = \sec b(-\alpha\beta u_x - \beta^2 u_y - \beta\gamma u_z + \alpha^2 v_x + \alpha\beta v_y + \alpha\gamma v_z) + \pi\alpha_1 X + \pi\beta_1 Y + \pi\gamma_1 Z; \tag{14}$$

$$\kappa\pi\mu_b = -\sec b[\alpha^2\gamma u_x + \alpha\beta\gamma u_y + \alpha\gamma^2 u_z + \alpha\beta\gamma v_x + \beta^2\gamma v_y + \beta\gamma^2 v_z \\ \alpha(\gamma^2 - 1)w_x + \beta(\gamma^2 - 1)w_y - \gamma(\alpha^2 + \beta^2)w_z] + \pi\alpha_2 X + \pi\beta_2 Y + \pi\gamma_2 Z. \tag{15}$$

3. A GLOBAL SOLUTION?

Although equations (13) to (15) can be used conjointly, should they be? In an important study, dated but still useful, Clube (1972) recommends against such a procedure. Because of observational error, as well as cosmic error dispersion (correlations caused by the velocity ellipsoid, for example), the variances in the equations are large, and one should make simplifying assumptions in S. Many, such as van de Kamp (1967), feel that, because of different systematic errors in each class of equation of condition, the equations should be solved separately. But Clube's recommendations were made when the observational data base exhibited relative paucity compared with the avalanche of data streaming in from such endeavors as the *Hipparcos* catalog. Nor was TLS, the original paper for which was only published in 1980, available as the reduction method. And although van de Kamp's suggestion merits consideration, it fails to take into account a significant aspect of the conjoint use of equations (13) to (15): the correct mathematical modeling of the relationships among the unknowns. Although w_x, w_y, w_z do not appear in equation (14), their influence will nevertheless be felt in that equation by the coupling caused by the solar velocity and some of the other unknowns, common to all of the equations. When the equations of condition are accumulated, either by forming normal equations of by use of orthogonal transformations, the resulting matrix becomes what is referred to in sparse matrix terminology as "doubly-bordered block diagonal". Performing separate solutions from the radial velocities, the proper motion in longitude, the proper motion in latitude, and subsequently statistically combining them, assumes that the borders are null, a demonstrably incorrect assumption; see Branham (1992a) for further discussion. Only a global solution for all of the parameters correctly models the underlying geometrical relationships. Rather than worry about systematic errors, it is more important to start the reduction with a valid mathematical model.

There also exists substantial feeling that solutions should only be performed on subsets of the data, such as spectrum-luminosity groups, for the putative reason that each subgroup exhibits different kinematical behavior. But the substantive reason arises from in the past using only OLS as the reduction technique. Because OLS assumes that the equations of condition are error-free, significant bias in the solution is only avoided by dividing the stars into groups of the same spectrum-luminosity class where the parallax is about the same for all of the stars in the group. To quote Smart (1967): "Indeed, progress can be made by assuming, for example, that the parallaxes are distributed closely around a value... which we may describe as a *mean parallax.*" Stars of the same spectrum-luminosity class follow closely this assumption. But when stars are grouped to give solutions for the various parameters, such as the Oort constants, and the solutions then statistically combined to yield a final value, one assumes, once again erroneously, the statistical independence of the subgroups. This usually invalid assumption nevertheless ameliorates partially, but only partially, the nefarious effects of the errors in the parallaxes.

That calculating solutions from subsets of the data, and then subsequently statistically combining them, constitutes at times a dubious procedure is manifested in the results from a previous study of mine, dealing with celestial mechanics rather than Galactic kinematics (Branham 1979). That study used 20,288 observations of five minor planets to estimate a number of parameters, including a possible equinox motion, $\Delta\dot{E}$, of the FK4 fundamental system. When only 19th century observations were used, I found $\Delta\dot{E} = 1.''237 \pm 1.''013$, and 20th century observations gave $\Delta\dot{E} = 1.''251 \pm 0.''469$. One would infer by the normal statistical combination of independent quantities that $\Delta\dot{E} = 1.''249 \pm 1.''226$. But a combined solution with all of the observations actually gave $\Delta\dot{E} = 0.''784 \pm 0.''208$.

It remains most likely true that kinematically different behavior does exist among different spectrum-luminosity groups, although Kurth (1967) feels that the assertion merits more rigorous statistical proof, but as a first step one should show that mixed TLS-LS yields reasonable results when applied globally to all of the data. Several advantages accrue to such a procedure: 1) because the distances run the gamut from that of Proxima Centauri to over 1 kpc, local variations should be smoothed over; 2) because of adequate all-sky coverage it should be possible to decouple the double wave in Galactic longitude in the proper motions and radial velocities caused by differential Galactic rotation from the single wave caused by the solar motion. (This consideration may not appear particularly relevant because the unknowns actually solved for in this paper do not include *explicit* $\sin 2l$ and $\cos 2l$ terms, but would become *very* relevant if one were to use the Oort-Lindblad rather than the Ogorodnikov-Milne model); 3) the greater distances for many of the proper motions assure that random motions do not dominate the genuine proper motions; 4) spectrum-luminosity groups are an idealization resulting from histogramming continuous properties into discrete groups, whereas a global solution treats the astrophysical properties as continuous; 5) one proceeds from a firm mathematical basis, the correct reduction model.

4. MIXED TLS-LS AND THE *HIPPARCOS* DATA

To apply equations (13)–(15) to Galactic kinematics, I used stars from the *Hipparcos* catalog, with its 113,710 stars with nonzero parallax. These stars have a median parallax of 4.790 mas, with a median error of 1.090 mas. This corresponds to a median distance of 209 pc with a variation, according to the median error, from 170 pc to 279 pc. Figure 1 shows the distribution of distances out to 1 kpc. The equations of condition, equations (13)–(15), were accumulated to form the system of equation (4) by the procedure given in Branham (1992b). Columns of the data matrix were scaled by the Euclidean norms of the same columns. No OLS solution was undertaken because I have shown previously (see Branham 1998) that OLS leads to poor, even wretched, results when used with Galactic kinematics and to calculate an OLS solution seems otiose.

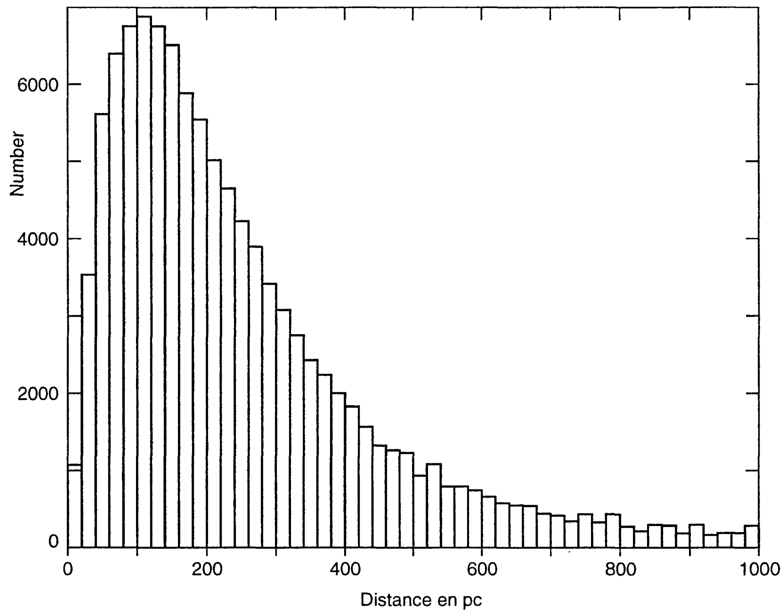

Fig. 1. Distribution of distances from *Hipparcos* catalog.

One may question whether column scaling alone is sufficient; the error in the distance, after all, increases with increasing distance. Perhaps the equations of condition should be weighted by a factor such as the parallax divided by the error of the parallax? In fact, such a weighting leads to *worse* results, probably because the high weight given to nearby stars with the lowest relative parallax errors leads to the solution being dominated by local irregularities. By "worse" I mean nonsense values for many of the unknowns, such as $A = 390$ km s^{-1} kpc^{-1}, $B = 245$ km s^{-1} kpc^{-1}, $K = -111$ km s^{-1} kpc^{-1}. The condition number of the equations of condition also increases to 387.

I tried and rejected another possible weighting scheme: remove from further consideration stars whose weight as defined in the previous paragraph is less than unity. These stars have an error in parallax larger than the parallax itself. With this criterion 3286 stars were rejected. But, perhaps surprisingly, the solution becomes, once again, unacceptable. The condition number increases to 387 (the equality of the condition number with that from the solution with the previous weighting scheme must be a fluke), and the values found for A, B, and K, among others, are once again nonsense. I can only speculate why this happens, but feel that the clue lies in the increased condition number, indicating that significant information has been excluded from the solution. The median distance of the *Hipparcos* stars is 209 pc, the median distance of the 3286 rejected stars is 704 pc, albeit with considerable error in the distance. The rejected stars, therefore, constitute a sample where local irregularities have little influence, but Galactic kinematic effects are pronounced, and their exclusion becomes pernicious.

Because I found no acceptable weighting scheme, each equation of condition entered with unit weight. It is interesting that TLS handles stars with large parallax error, such as those of the previous paragraph, with no difficulty and calculates an acceptable solution.

Not all of the data from the *Hipparcos* catalog can be accepted. Known multiple stars, flagged in the catalog, contaminate the proper motion by confusing orbital motion with genuine proper motion and should be excluded, and some of the *Hipparcos* solutions for the astrometric data in the catalog are substandard ($\chi^2 > 3$), also flagged in the catalog, and should likewise be excluded. Stars belonging to the Gould belt should also be excluded because they will hardly share the same kinematics as those belonging to the Galactic belt. The next sections discuss how the Gould belt stars were excluded. Upon doing this I remain with 98,269 proper motions in each coordinate (the proper motions are transformed from right ascension and declination to Galactic latitude and longitude by use of the standard expressions). Table 1 shows the breakdown of the observations according to spectral type.

TABLE 1

BREAKDOWN OF THE OBSERVATIONS BY SPECTRAL TYPE

Spec. Type	Percent.	Spec. Type	Percent.
O	0.09	M	4.26
B	4.87	R	0.06
A	16.17	N	0.03
F	22.10	S	0.01
G	20.18	C	0.12
K	29.62	Other	2.49

TABLE 2

SINGULAR VALUE ANALYSIS OF EQUATIONS OF CONDITION

Unknown	Singular Value	Percent.
u_x	3583.64	32.34
u_y	3170.61	28.63
u_z	3110.01	28.08
v_x	53.45	0.48
v_y	178.49	1.61
v_z	169.53	1.53
w_x	166.17	1.50
w_y	140.83	1.27
w_z	135.08	1.22
X	132.59	1.21
Y	117.00	1.07
Z	116.40	1.06

The *Hipparcos* catalog itself, being one of astrometric data, does not contain the radial velocities needed in equation (13). To fill this lacuna I took radial velocities from the Wilson catalog (Nagy 1991) and from the Barbier-Brossat-Petit catalog (1994), which yielded 8613 radial velocities. There are far fewer radial velocities than proper motions, but their inclusion seems warranted because, although numerically inferior, they nevertheless represent a large number compared with what was used in many pioneering studies on Galactic kinematics and, furthermore, introduce no new unknowns into the system. That their inclusion is not deleterious can be seen by calculating the singular values of the 205,151 equations of condition, before scaling, (98,269 in Galactic longitude, 98,269 in Galactic latitude, and 8613 in radial velocity), which shows that all of the unknowns are well determined and that the condition number of the linear system, 67.0, is low; see Table 2.

5. SELECTION OF GOULD BELT STARS

Stars belonging to the Gould belt should be excluded because they will hardly share the same kinematic properties as the Galactic belt stars. It is usually felt that only the bright OB stars belong to the Gould belt, although some authors, such as Fresneau et al. (1996), present evidence that other spectral types may also pertain to the belt. To check which spectral types are likely candidates for inclusion in the Gould belt, I performed a standard test, the runs test (Wonnacott & Wonnacott 1972), for randomness in Galactic latitude.

One assumes that Galactic belt stars will be randomly distributed in b whereas the Gould belt stars will not. The runs test showed that spectral classes A-M are indeed randomly distributed, at least roughly. (Perfect concordance cannot be expected because of possible selection effects). Table 3 shows that the OB stars, on the other hand, show a clear break in randomness near visual magnitude 6.5.

The procedure I followed to identify the Gould belts stars is similar to that of Stothers & Frogel (1974): pass a plane through the bright OB stars and another through the fainter OB stars. An individual star is considered a Gould belt star if it is closer to the former. The procedure is iterated. I also applied a criterion in distance: a star has to be nearer than 500 pc to be a possible Gould belt candidate. After the final iteration 3630 OB stars were classified as Gould belt and 4914 as Galactic belt. Figure 2 shows the Gould belt plane, with a noticeable inclination with respect to the Galactic plane. After this final iteration the Gould belt stars exhibited 890 runs out of an expected 1815, whereas the Galactic belt stars exhibited 2266 runs out of an expected 2457. Although there are fewer runs than expected, the discrepancy is far less than with the stars classified as Gould belt. Furthermore, Miyamoto & Zi (1998) present evidence that Galactic belt OB stars exterior to the Sun are inclined slightly with respect to the Galactic plane. Thus, perfect randomness may be unachievable. It thus appears as if the classification regarding Gould belt versus Galactic belt appears sound. The l and b values found for the pole of the Gould belt, $l = 211.^\circ785$, $b = 79.^\circ755$, although different from those of the pole calculated for the Galactic belt, $l = 4.^\circ752$, $b = 88.^\circ789$, are nevertheless, not too close to the values that Gould originally found, $l = 201^\circ$, $b = 72^\circ$. Gould, however, considered only bright stars. My solution includes in the Gould belt fainter OB stars: of the 3630 Gould belt OB stars the median m_v is 7.55 versus 8.02 for the Galactic belt; likewise, the maximum magnitude is 0.18 versus 0.98. The runs test shows that the discrimination between Gould and Galactic belt OB stars is reasonably firm. The 3630 Gould belt OB stars were excluded from further consideration. The final conclusions, however, are fairly insensitive to the inclusion or exclusion of the Gould belt OB stars, as I found by experimentation.

6. OUTLIERS

TLS is even more sensitive to outliers than is OLS. Before reliable solutions can be calculated, therefore, a criterion for outlier rejection must be developed. But this is tricky because most outlier rejection criteria, such as Pierce's (Branham 1990) assume a normal distribution for the residuals. But are the residuals normally distributed? In many instances, no. To examine the distribution I decided to calculate a first solution from a norm minimization procedure far less influenced by outliers than the normal distribution, the L_1 criterion. But because of error in the equations of condition one cannot use the standard L_1 algorithm, given in Branham (1990), which, like OLS, assumes that the data matrix is error-free. One may, however, calculate an orthogonal L_1 solution by a norm minimization technique:

TABLE 3

RUNS TEST FOR DIFFERENT SPECTRAL TYPES

Spec. Type	Expected Runs	Actual Runs
OB ($m_v < 6.5$)	704	285
OB ($m_v > 6.5$)	3,568	3,217
A	7,946	7,982
F	10,927	10,847
G	10,043	9,833
K	14,945	14,248
M	2,168	2,173

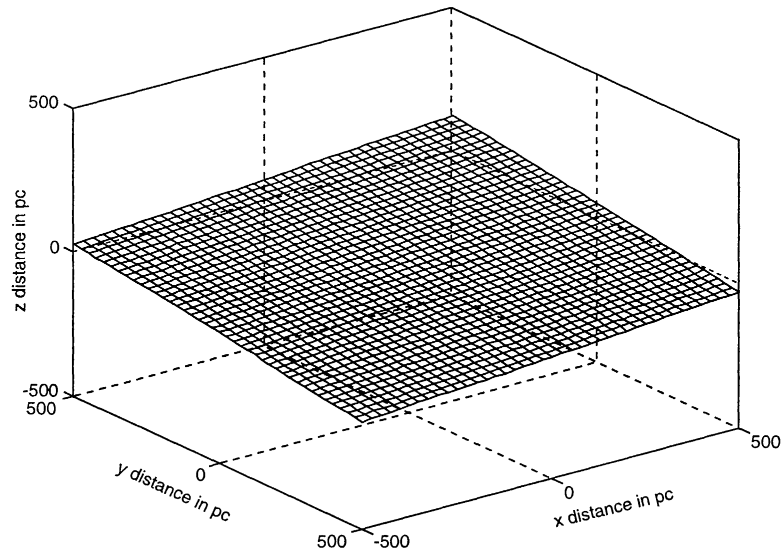

Fig. 2. Plane defined by Gould Belt OB Stars.

$$\frac{\| A \cdot x - d \|_1}{\sqrt{1 + x_E^T \cdot x_E}} = min, \tag{16}$$

where x is the solution and $x_E^T \cdot x_E$ refers to the scalar product of the unknowns corresponding to error columns in A (the unknowns for the solar velocity). For the theory behind equation (16) see Späth & Watson (1987). Upon calculating a solution from equation (16) and examining the residuals, I find that they represent a distribution, shown in Figure 3, far from the normal, a narrower, more heavy tailed distribution. Use of Pierce's criterion, therefore, seems unjustified, and I chose to use five times the mean absolute deviation (MAD) as the cutoff for an acceptable residual. Use of the MAD rather than the mean error of unit weight, $\sigma(1)$, seems indicated because of the behavior of the distribution. This criterion eliminated 3.57% of the equations, a modest trim although higher than what would be given by Pierce's criterion, 0.1%. This amount of trim is lower than what some feel is reasonable for long-tailed, Cauchy type distributions, where a 10% to even 25% trim would be acceptable —see Stigler (1977). Miyamoto & Sôma (1993), for example, use, depending on the solution, a trim of from 12% to 18%. Stigler, however, finds that, in general, extreme trimming gives results no better, or even worse, than more moderate trimming. Here, moreover, part of the long tail undoubtedly comes from high velocity stars, genuine data rather than outliers. About 3400 of the stars should be high velocity. The trim selected will eliminate about 100 of these, statistically insignificant given the total number of observations, but will also eliminate genuine outliers. A higher trim would eliminate most of the high velocity stars, difficult to justify on statistical or any other criteria.

But such a sparse trim as that given by Pierce's criterion can hardly be justified either, as can be seen by looking at a plot of the sorted absolute values of the residuals, Figure 4. The spike caused by the heavy tails of Fig. 3 is more than noticeable. Although high-velocity stars undoubtedly contribute to the spike, a nefarious effect is also likely at work: spurious proper motions. Because they are based on a short time span, about three years, the *Hipparcos* proper motions can confuse orbital motion of unknown long period multiple stars with a genuine proper motion. These stars will not be flagged as multiple in the catalog. Wielen et al. (1999) have given a name, "cosmic error", to this phenomenon. That cosmic error is probably present and can be inferred if we compare the spike in Fig. 4 with that in Fig. 2 of Branham (1998), based on the Yale Bright Star Catalog, where the spike is still present, but is much less pronounced. Confirming evidence comes from a comparison between the Astrographic Reference Star Catalog (ACRS) (Corbin, Urbain, & Warren 1991) proper motions and the *Hipparcos* proper motions. Although the ACRS proper motions, incorporate mean errors of the order of 4–5 mas yr^{-1} compared with *Hipparcos*'s 1 mas yr^{-1}, they also cover a time span ten times longer and

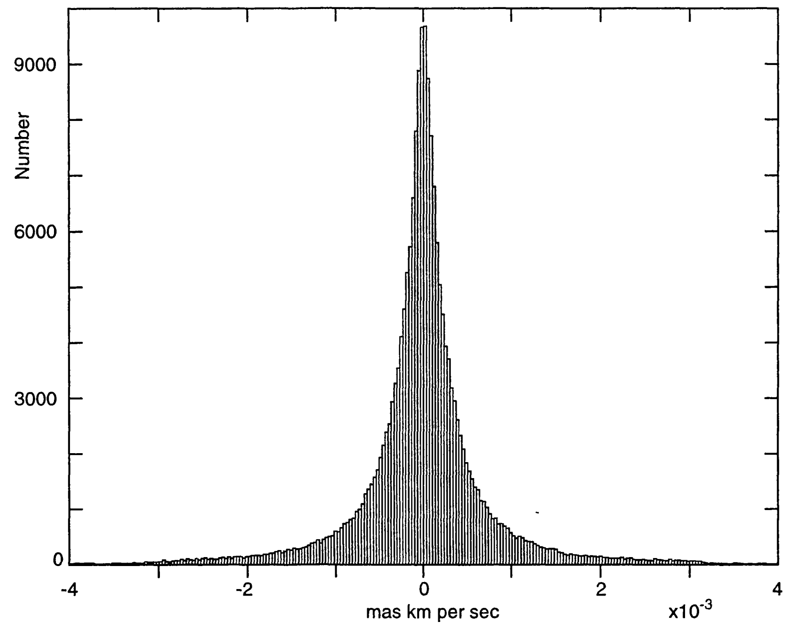

Fig. 3. Distribution of the residuals.

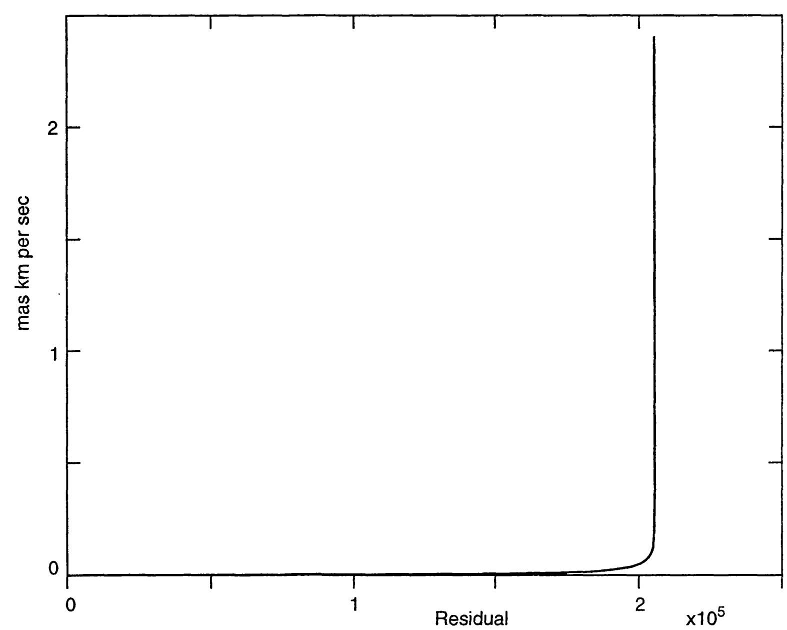

Fig. 4. Sorted absolute value of the residuals.

should, therefore, be less influenced by cosmic error. By using the BD and CoD numbers in both catalogs, I found 79% of the *Hipparcos* stars in the ACRS. Although the median difference in Galactic longitude of the stars in common is only −0.06 mas, some large differences occur, such as the maximum difference of 6187 mas; in Galactic latitude the corresponding numbers are 0.37 mas and 1380 mas. To minimize the cosmic error one should therefore use a reasonable, but not excessive, trim.

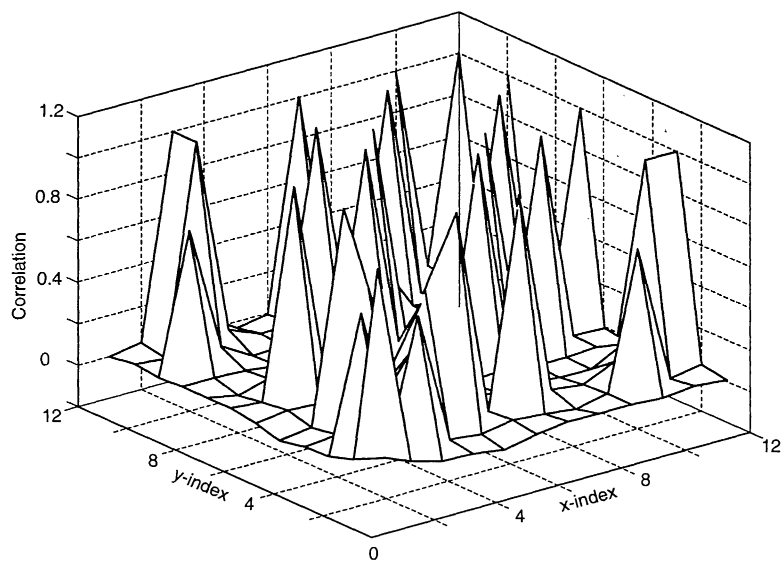

Fig. 5. Surface plot of the correlations.

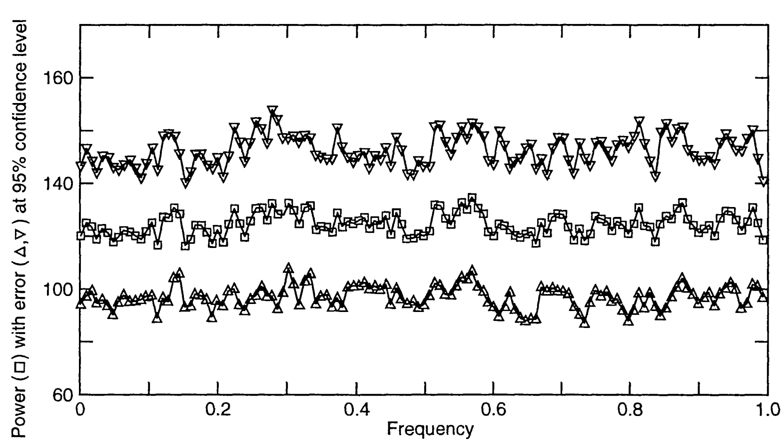

Fig. 6. Spectral analysis of the residuals.

7. THE SOLUTIONS

I calculated various solutions, but will only show, in Tables 4 and 5, the ones accepted. The mean errors come from the procedure that Branham (1999) gives for a covariance matrix for TLS; although that procedure also allows for heteroscedastic data, the data here are purely homoscedastic. The 12 $\times$ 12 correlation matrix is too large to exhibit as a table and is shown instead as the surface plot in Figure 5. Only three correlations, aside from the 100% correlation of each variable with itself, exceed the 50% level (the three peaks in the plot):

64.0% between u_y and u_z; 66.2% between u_y and X; and 67.3% between u_z and v_x. None of these correlations, although significant, is high, nor is the condition number of the data matrix, 67.0 for the unscaled matrix and 2.71 for the column-scaled matrix, unreasonable. Thus, all of the unknowns appear well determined.

Not only well determined, but the residuals from this solution show no systematic tendencies, such as what might be caused by the velocity ellipsoid, although they hardly represent a normal distribution. This most likely occurs because the equations of condition are weighted by π, thus eliminating correlations caused by the velocity ellipsoid. Statistics on the residuals, see Table 6, show significant deviations from a normal distribution.

The actual distribution is longer-tailed, as measured by Hogg's Q factor, than the normal, somewhat skewed, and much more peaked (leptokurtic), as measured by the kurtosis, than the normal. Although the residuals differ from a normal distribution, they are random, as determined from the standard runs test given in most statistics texts. Out of 197,835 residuals, there are slightly more runs, 100,309, than expected, 98,918, but not so overwhelmingly more as to exhibit systematic tendencies. Spectral analysis of the residuals, shown in Figure 6, reveals no systematic tendencies nor indication of possible periodicities, as is confirmed by a χ^2 test with 95% confidence interval.

If we assume that the motion in a system is parallel to the z-plane, then $w_x = w_y = w_z = 0$. If, furthermore, the plane $z = 0$ is a plane of symmetry, then one can find expressions for the Oort A and B constants and for a K term (Ogorodnikov 1965)

$$A' = (u_y + v_x)/2; \tag{17}$$

$$C' = (u_x - v_y)/2; \tag{18}$$

$$B = (-u_y + v_x)/2; \tag{19}$$

$$K = (u_x + v_y)/2. \tag{20}$$

The Oort constant A follows upon defining

$$\begin{aligned} A' &= A\cos 2l_1 \\ C' &= -A\sin 2l_1, \end{aligned} \tag{21}$$

where l_1 is the direction defining the longitude of the Galactic center. From equation (21) we have

$$\begin{aligned} A &= \sqrt{A'^2 + C'^2}\ , \\ l_1 &= \frac{1}{2}\arctan(-C'/A')\,. \end{aligned} \tag{22}$$

The solar velocity, S_0, comes from $(S_x^2 + S_y^2 + S_z^2)^{1/2}$. The K term of equation (20) is more complicated than representing simply a constant offset in the equations of condition for radial velocity with units of km s^{-1}; here its units are mas km s^{-1}, a variable that gives a constant value when multiplied by the distance to a star, and it appears in both the equation of condition for radial velocity and for proper motion in Galactic latitude.

Table 5 shows various quantities, and their mean errors, deduced from equations (6), (17) to (22). Rices's procedure (1902) calculates the mean errors from the mean errors in Table 4 and the covariance matrix; for the unknowns assumed independent, S_0, A, and l_1, the covariances are set to zero.

8. DISCUSSION

Both Tables 4 and 5 give reasonable results. l_1 differs not greatly from the accepted direction to the center of the Galaxy. The solar velocity is in the range 20–30 km s^{-1} found by others, and the direction of the solar motion lies towards $l = 61.^\circ 69$, $b = 18.^\circ 1$, similar to what others have determined. Miyamoto & Sôma (1993), using K and M giants, find $l = 59.^\circ 7$, $b = 23.^\circ 8$. My previous study (Branham 1998) gave

TABLE 4

THE SOLUTION FOR TWELVE UNKNOWNS

Unknown	Value and Mean Error (mas km s^{-1})
u_x	.76921D+01 ± .11549D+01
u_y	.21677D+02 ± .61578D+00
u_z	−.10260D+02 ± .76412D+00
v_x	−.26808D+00 ± .64333D+00
v_y	.50096D+01 ± .11542D+01
v_z	.39386D+00 ± .76382D+00
w_x	−.57461D+00 ± .58568D+00
w_y	−.16640D+01 ± .56081D+00
w_z	.37461D+01 ± .11670D+01
X	−.10300D+02 ± .60561D−01
Y	−.19125D+02 ± .54771D−01
Z	−.70920D+01 ± .44763D−01
$\sigma(1)$	0.10897 mas km s^{-1}
Nr. Obsns.	197,835

TABLE 5

SOLUTION FOR OTHER UNKNOWS

Unknown	Value and Mean Error	Unknown	Value and Mean Error
S_0	22.850 ± 0.055 km s^{-1}	K	6.351 ± 0.809 km s^{-1} kpc^{-1}
A	10.788 ± 0.495 km s^{-1} kpc^{-1}	ω_x	−0.118 ± 0.104 mas yr^{-1}
B	−10.973 ± 0.488 km s^{-1} kpc^{-1}	ω_y	−1.124 ± 0.141 mas yr^{-1}
l_1	$-3.^\circ 571 \pm 0.^\circ 805$	ω_z	−2.314 ± 0.092 mas yr^{-1}

TABLE 6

STATISTICS OF THE RESIDUALS (REAL, UPPER; NORMAL, LOWER)

Median Residual (mas km s^{-1})	Mean Residual (mas km s^{-1})	Q Factor	Skewness	Kurtosis
−0.46195e–05	−0.56257e–05	0.4190	−0.0846	5.6857
0.0	0.0	2.54	0.0	0.0

$l = 63.°3$, $b = 26.°3$. The A and B constants also lie in range given by other determinations, although their near equality is somewhat surprising. This, however, may be a consequence of the greater distances to which the trigonometric parallaxes now reach. Kerr & Lynden-Bell (1986), who surveyed the various determinations that lead to the IAU recommended values of 14 km s^{-1} kpc^{-1} for A and -12 km s^{-1} kpc^{-1} for B, remark that equal values of 13 and -13 km s^{-1} kpc^{-1} become more consistent witha flat rotation curve for the Galaxy given by radioastronomical observations, made at greater distances than optical observations. A and B yield a Galactic rotational velocity near the Sun of $V_\theta = 185$ km s^{-1} if we take the distance to the Galactic center R_0 as 8.5 kpc. Although lower than the IAU recommended 220 ± 20 km s^{-1}, this also agrees with what some others have found. My 1998 study (Branham 1998), for example, although containing only 5% of the stars of the present study, but also a mixture of spectral and luminosity classes, found $V_\theta = 171$ km s^{-1}, and Miyamoto & Sôma (1993) in a solution based on only K and M giants determine 177 km s^{-1}. Zhu & Yang (1999), on the other hand, also using K and M giants, find $V_\theta = 249.6$ km s^{-1}. This study and Zhu and Yang's use *Hipparcos* proper motions, my previous study and Miyamoto and Sôma's systems are based on the FK5; the dispersion from 171 through 249.6 km s^{-1} permits no easy association of results versus the reference system or spectrum-luminosity class.

The waters are further muddied by Miyamoto & Zi's study (1998), which used only *Hipparcos* proper motions of OB stars and found $V_\theta = 268.7$ km s^{-1}, also disagreeing with the IAU recommendation but in the other direction and 84 km s^{-1} higher than the value found in this study. Because Miyamoto and Zi use only OB stars some discrepancy with my value determined from a mixture of all of the spectral types is inevitable, but the size of the difference is surprising. And because we both use *Hipparcos* proper motions the reason that they give for the disparity between their value and the IAU recommendation, differences in the proper motion system between the *Hipparcos* catalog and ground-based catalogs, becomes inapplicable. Nor can the discrepancy be attributed easily to trimming the data. Unlike Miyamoto & Sôma (1993), Miyamoto and Zi use a parsimonious trimming philosophy similar to mine that rejects only 2.4% of the stars. Although I incorporate radial velocities and they do not, it is difficult to see how an 84 km s^{-1} difference can be explained by an addition of 4% of equations of condition corresponding to radial velocity to the total number of equations of condition. One possibility for explaining the discrepancy arises from the formation of the equations of condition: they include stars out to a distance of 3 kpc, three times the effective upper limit of the distances I use, but is it legitimate to employ Miyamoto and Zi's equation (1), a first-order Taylor expansion, at such large distances? Would it not be advisable to use second-order expansions beyond 1 kpc? They refer to solving the equations of condition by "generalized least squares". Are they using the term in the sense in which it is used in numerical analysis, where "generalized least squares" (GLS) means calculating a least squares solution by incorporation of a given covariance matrix of the observations? (See Björck (1996) for further discussion). If so, the calculated GLS solution depends on the given covariance matrix. It would be useful to know what this covariance matrix is. But if they merely use "generalized least squares" as a euphemism for least squares, then is it legitimate to apply least squares to a problem where the distances vary from 0.1 kpc to 3 kpc? Although they use spectroscopic parallaxes, which do not exhibit the feature of increasing error in the distance as the distance increases, although as explained previously this feature does not seem to derail TLS, the equations of condition will still contain at least some error. Would it not be preferable to use TLS? The resolution of the conundrum of why the values of V_θ vary so much among different studies undoubtedly lies in the careful consideration of these, and other, questions, a far from trivial undertaking.

The K term, if evaluated at the median distance of the *Hipparcos* parallaxes, yields 1.33 km s^{-1}, again reasonable considering that the term has been found significant only for early stars (5.3 km s^{-1} for O and B stars and 1.4 km s^{-1} for A stars). Given that 21% of the stars are O, B, and A one can hardly complain about the value determined.

The z component of the rotation vector ω is the same as the B constant and needs no further comment. ω_x and ω_y, subject to various interpretations in terms of an equinox motion, corrections to lunisolar and planetary precession, and others, although small still exceed the supposed limits of the inertiality of the *Hipparcos* system, $\pm$ 0.25 mas yr^{-1}. But the results give what they give. That a value close to 1 mas yr^{-1} seems not merely a fluke is reinforced upon our doing a slightly different solution. One way of simplifying the tensor S, used in many studies, *assumes* that the diagonal elements are zero. Given the singular values from the data matrix for the data presented in this paper, such an assumption scarcely merits credibility and renders nugatory any solution based on it. But *if* we should nevertheless make the assumption, then we would obtain the values (without showing mean errors) for the unknowns that can be determined without the diagonal elements: $S_0 = 22.860$ km s^{-1}; $A = 12.479$ km s^{-1} kpc^{-1}; $B = \omega_z = -11.027$ km s^{-1} kpc^{-1}; $\omega_x = -0.149$ mas yr^{-1}; $\omega_y = -1.101$ mas yr^{-1}. The solar motion remains nearly the same, V_θ increases to 200 km s^{-1}, still lower than the IAU recommendation, the near equality between A and B disappears, and ω_x and ω_y remain nearly the same. Thus, with or without the diagonal elements the rotation tensor leads to rotations higher than the supposed stability limit of the *Hipparcos* system. This *may* be a consequence of remaining cosmic error in the proper motions. To test the supposition one could combine the *Hipparcos* proper motions with those from a ground based system, by using Wielen et al's procedure (1999) for example. That is, however, a nontrivial task and best left for further research. One cannot, however, exclude entirely the possibility of higher stability limits for the *Hipparcos* system than those published. Pinsonneault et al. (1998), for example, give evidence for *systematic* errors in the *Hipparcos* parallaxes at the 1 mas level, higher than what one would infer from the catalog description.

9. CONCLUSIONS

The *Hipparcos* parallaxes and proper motions, along with ancillary radial velocities, yield acceptable values for the parameters of Galactic kinematics when one uses the Ogorodnikov-Milne model and reduces the data with mixed TLS-LS. *All* of the unknowns are well determined as measured by their singular values. Only the x and y components of the rotation tensor appear slightly anomalous, although this *may* be caused by remaining cosmic error in the proper motion system.

Future research should investigate the components of the rotation tensor by combining the *Hipparcos* proper with ground-based proper motions and should also look into possible differences caused by different spectrum-luminosity groups. Because the *Hipparcos* catalog does not include luminosity classes, although it does incorporate spectral classification, that information would have to come from another source, perhaps the SKYMAP Catalog, electronic address fttp://adc.gsfc.nasa.gov/pub/adc/archive/catalogs/5/5102. Both of these endeavors, including ground-based proper motions and luminosity classes, represent nontrivial extensions of the work presented here.

I would like to dedicate this article to the memory of a colleague, friend, and a great astrometrist, Dr. Heinrich K. ("Heinz") Eichhorn.

REFERENCES

Barbier-Brossat, M., Petit, M., & Figon, P. 1994, A&AS, 108, 603
Björck, A. 1996, Numerical Methods for Least Squares Problems (Philadelphia: SIAM), 162
Branham, Jr., R. L. 1979, Astronomical Papers of the American Ephemeris and Nautical Almanac, 21, Part 3
________. 1989, Computers in Physics, 3, 42
________. 1990, Scientific Data Analysis (New York: Springer), 78, Sec. 6.4
________. 1992a, AJ, 104, 1658
________. 1992b, in Numerical Methods in Engineering and Applied Sciences Part 2, eds. H. Alder, J. C. Heinrich, S. Lavanchy, E. Oñate, & B. Suarez (Barcelona: CIMNE), 952
________. 1995, Celest. Mech., 61, 239
________. 1997, in Recent Advances in Total Least Squares Techniques and Errors-in-Variables Modeling, ed. S. Van Huffel (Philadelphia: SIAM), 371
________. 1998, Ap&SS, 257, 107
________. 1999, AJ, 117, 1942
Clube, S. V. M. 1972, MNRAS, 159, 289
Corbin, T. E., Urbain, S. E., & Warren, W. H. 1991, Astrographic Catalog Reference Stars, CD-ROM Version (Lanham, Md.: ST Systems Corp.)
Fresneau, A., Acker, A., Jasniewicz, G., & Piat, M. 1996, AJ, 112, 1614
Golub, G. H., & Van Loan, C. F. 1980, SIAM Jour. Numer. Anal., 17, 883
Kerr, F. J., & Lynden-Bell, D. 1986, MNRAS, 221, 1023
Kurth, R. 1967, Introduction to Stellar Statistics (Oxford: Pergamon Press), Sec. 21
Miyamoto, M., & Sôma, M. 1993, AJ, 105, 691
Miyamoto, M., & Zi, Z. 1998, AJ, 115, 1483

Nagy, T. A. 1991, Documentation for the Machine-Readable Version of the Wilson General Catalogue of Stellar Radial Velocities, CD-ROM Version, (Lanham, Md.: ST Systems Corp.)
Ogorodnikov, K. F. 1965, Dynamics of Stellar Systems (Oxford: Pergamon Press), pp. 61-63, 74-75
Pinsonneault, M. H., Stauffer, J., Soderblom, D. R., King, J. R., and Hanson, R. B. 1998, ApJ, 504, 170
Rice, H. L. 1902, AJ, 22, 149
Smart, W. M. 1967, Stellar Kinematics (New York: Wiley), Sec. 8.42
Smith, H. Jr., & Eichhorn, H. 1996, MNRAS, 281, 211
Späth, H., & Watson, G. A. 1987, Numer. Math., 51, 531
Stigler, S. M. 1977, Annal. Stat., 5, 1055
Strothers, R., & Frogel, J. A. 1974, AJ, 79, 456
Van Huffel, S., & Vandewalle, J. 1991, The Total Least Squares Problem: Computational Aspects and Analysis (Philadelphia: SIAM), Sec. 3.6.2, 270
Van de Kamp, P. 1967, Principles of Astrometry (San Francisco: Freeman), 69
Wielen, R., Schwan, H., Dettbarn, C., Jahreiss, H., & Lenhardt, H. 1999, in Modern Astrometry and Astrodynamics, ed. R. Dvorak, H. F. Haupt, & K. Wodnar (Viena: Österreichischen Akademie der Wissenschaften), 161
Wonnacott, T. H., & Wonnacott, R. J. 1972, Introductory Statistics, 2nd. Ed. (New York: Wiley), 409
Zhu, Z., & Yang, T. 1999, AJ, 117, 1103

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