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A COMPARISON OF GEOMETRICAL AND ANALYTICAL METHODS OF ASTROMETRIC CORRECTIONS

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RESUMEN

En el siguiente trabajo se presentan y relacionan mediante un método de mínimos cuadrados dos modelos diferentes, uno geométrico y otro analítico, para la corrección de sistemas de referencia. La corrección mínimo-cuadrática al sistema de referencia inducida por el método analítico se compara con la que proporciona directamente el método geométrico que fue contrastado previamente obteniendo buenos resultados en el sistema fundamental FK5.

El método geométrico proporciona la orientación relativa del FK5 con la referencia dinámica dada por una teoría planetaria, en tanto que el analítico es más adecuado para la corrección de los errores dependientes de zona.

ABSTRACT

A geometrical and an analytical model of Reference System correction are presented and connected by means of a simple least squares method. The correction induced by the analytical method is compared with that obtained directly through the geometrical method which was previously tested in the FK5 system. The geometrical method determines the relative orientation of the FK5 with respect the dynamical reference frame given by the Planetary Theory, while the analytical method is more suitable to correct the error depending on the zone.

Key Words: **ASTROMETRY — CATALOGS — MINOR PLANETS — REFERENCE SYSTEMS**

1. INTRODUCTION

The first attempt to employ minor planets to correct the reference system was proposed in the beginning of this century in the classical work of Numerov, an astronomer from the ITA (Institute of Theoretical Astronomy) from St. Petersburg. He proposed an ambitious observational program to obtain data of selected asteroids whose orbits lie between $-20°$ and $+20°$ of declination. For historical reasons, this program was never accomplished but it set the basis to develop new treatments to correct the reference system. Basically, these treatments consist on the development of the corrections to a catalogue in a suitable set of functions in the form,

$$\Delta\alpha = \sum^{N} \Delta\xi_n F_n(\alpha,\delta),$$
$$\Delta\delta = \sum^{N} \Delta\eta_n F_n(\alpha,\delta),$$

where $\Delta\xi_n$ and $\Delta\eta_n$ are coefficients that should be computed for each case and while the choice of the functions F_n is different according to authors. For example, Duma (1991) considers simple developments in sines and cosines taking $N=2$ while Batrakov takes Chebysheff polynomials also with $N=2$ (see Batrakov et al. 1999).

At about the same time than Numerov, Brouwer suggested a similar observational program involving 14 asteroids in a zone of $-30° \leq \delta \leq 30°$. Pierce (1971) revised this project and measured, and reduced the plates

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again. After an analysis of the observational positions minus the calculated positions (O-C), he obtained the systematical corrections for the little planets, together with a correction to the Equator and the Ecliptic of the Yale catalogs.

In the year 1976 the IAU adopted an observational program of 20 minor planets spanning over the period 1976–1990. One of the main objectives of this program was the determination of the zero point for the FK4 catalog. To this aim, data of the asteroids covering a band of 30° around the Ecliptic were taken. About 35 observatories all around the world, took part in this project which has a continuation in a new program of observation of selected minor planets covering the years 1991–2000. A wider historical revision may be found in Batrakov et al. (1999).

In this paper two different methods to correct the reference system using minor planets are presented. The so called geometrical method is explained in § 2. It provides the corrections to the reference system and the minor planet elements through of the application of a geometrical correction obtained by means of an infinitesimal rotation matrix which provides the corrections to the reference system. Secondly, in the analytical method exposed in § 3 we consider a development of the O-C in a suitable set of orthogonal functions to obtain an analytical expression for the residual function in order to obtain directly corrections to the considered reference frame.

The geometrical method gave numerical results that were contrasted with the FK4 (Marco, López, & Martínez [1996]) but the analytical method provides a more general method that must be contrasted. To this aim, we compare the rotations induced in the least squares sense by the last model with the parameters obtained from the geometrical adjustment and, taking into account that the FK4 system is no more in use, we shall do all the calculations in the FK5 system.

The first step consists in the application of the corrections $\left\{\overrightarrow{\Delta\sigma^\delta}\right\}$ to the initial elements $\overrightarrow{\sigma^\delta}$ to the calculated right ascension and declination for whatever time t

$$\alpha\left(\overrightarrow{\sigma^\delta}+\overrightarrow{\Delta\sigma^\delta};t\right) \text{ and } \delta\left(\overrightarrow{\sigma^\delta}+\overrightarrow{\Delta\sigma^\delta};t\right). \tag{1}$$

Both methods are developed taking into account the perturbation elements technique Marco, López, & Martínez (1997), which is briefly summarized in Appendix A, in order to consider the longest time of integration possible and also to use observations as old as possible.

2. GEOMETRICAL APPROACH

Let us denote as $\overrightarrow{X_c}$ the unitary vector pointing to the heliocentric equatorial position (α, δ) and let $\overrightarrow{X_F}(\alpha-\Delta\alpha, \delta-\Delta\delta)$ be the same vector after a geometrical correction made by means of the infinitesimal rotation matrix,

$$R = \begin{bmatrix} 1 & \Delta\xi & \Delta\eta \\ -\Delta\xi & 1 & \Delta\varepsilon \\ -\Delta\eta & -\Delta\varepsilon & 1 \end{bmatrix}, \tag{2}$$

being $\Delta\xi, \Delta\eta$, and $\Delta\varepsilon$ three infinitesimal rotation around the main axes of the reference system. The following relation between the two vectors is applied,

$$\overrightarrow{X_F}(\alpha-\Delta\alpha, \delta-\Delta\delta) = R\overrightarrow{X_c}(\alpha, \delta), \tag{3}$$

taking into account the small-angle approximations for the right ascension,

$$\begin{aligned} \cos(\alpha-\Delta\alpha) &= \cos\alpha + \Delta\alpha\sin\alpha, \\ \sin(\alpha-\Delta\alpha) &= -\Delta\alpha\cos\alpha + \sin\alpha, \end{aligned}$$

and equally for the declination, the following system appears,

$$\begin{aligned}\Delta\xi \sin\alpha\cos\delta + \Delta\eta\sin\delta &= \Delta\alpha\sin\alpha\cos\delta + \Delta\delta\cos\alpha\sin\delta, \\ -\Delta\xi\cos\alpha\cos\delta + \Delta\varepsilon\sin\delta &= -\Delta\alpha\cos\alpha\cos\delta + \Delta\delta\sin\alpha\sin\delta, \\ \Delta\eta\cos\alpha\cos\delta + \Delta\varepsilon\sin\delta &= \Delta\delta\cos\delta.\end{aligned} \quad (4)$$

Solving (4) these incremental values are obtained,

$$\Delta\alpha = \Delta\xi + \Delta\eta\sin\alpha\tan\delta - \Delta\varepsilon\cos\alpha\tan\delta, \quad (5)$$

$$\Delta\delta = \Delta\eta\cos\alpha + \Delta\varepsilon\sin\alpha,$$

which provide the individual residuals,

$$\Delta\alpha_i^r = (\alpha_i^r)^{obs} - \alpha_i^r(\overrightarrow{\sigma_r^\circ} + \overrightarrow{\Delta\sigma_r^\circ}; t_i^r) + \Delta\xi + \Delta\eta\sin\alpha_i^r\tan\delta_i^r - \Delta\varepsilon\cos\alpha_i^r\tan\delta_i^r, \quad (6)$$

$$\Delta\delta_i^r = (\delta_i^r)^{obs} - \delta_i^r(\overrightarrow{\sigma_r^\circ} + \overrightarrow{\Delta\sigma_r^\circ}; t_i^r) = \Delta\eta\cos\alpha_i^r + \Delta\varepsilon\sin\alpha_i^r,$$

for each minor planet r and each observation i. So, the residual function arises,

$$R^2(\overrightarrow{\Delta\sigma_1^\circ}, ..., \overrightarrow{\Delta\sigma_{N_a}^\circ}; \overrightarrow{\Delta x}) = \sum_{r=1}^{N_a}\sum_{i=1}^{n_r}(R_i^r)^2 = \sum_{r=1}^{N_a}\sum_{i=1}^{n_r}\left[(\Delta\alpha_i^r)^2\cos^2\delta_i^r + (\Delta\delta_i^r)^2\right], \quad (7)$$

where $\overrightarrow{\Delta x} = [\Delta\xi, \Delta\eta, \Delta\varepsilon]^t$, N_a is the number of asteroids employed in the adjustment and n_r the number of observations for the r^{th} asteroid. The application of the minimum condition to (7) provides the normal system

$$\begin{bmatrix} S_1 & 0 & . & 0 & T_1^1 \\ 0 & . & . & 0 & . \\ . & . & . & 0 & . \\ 0 & . & 0 & S_{N_a} & T_{N_a}^1 \\ (T_1^1)^t & . & . & (T_{N_a}^1)^t & F^1 \end{bmatrix} \begin{bmatrix} \overrightarrow{\Delta\sigma_1^\circ} \\ \\ \\ \overrightarrow{\Delta\sigma_{N_a}^\circ} \\ \overrightarrow{\Delta x} \end{bmatrix} = \begin{bmatrix} \overrightarrow{W_1} \\ \\ \\ \overrightarrow{W_{N_a}} \\ \overrightarrow{U_1} \end{bmatrix} \quad (8)$$

that is solved through the groups method following Levallois (1969). The expressions appearing in the matrix of (8) are detailed in Appendix B.

3. ANALYTICAL SIGNAL PROCESSING

As stated in § 1, in this method our aim is to develop the O-C residual function in suitable orthogonal functions. The orbital motion of the considered minor planets is contained in a band around the Ecliptic so the classical spherical harmonics are not suitable functions for this analytical adjustment. That is why, following Bien et al. (1978), we consider the new set of spherical functions,

$$K_{n,m}(\lambda,\beta) = P_n\left(\frac{\sin\beta_{\max} + \sin\beta_{\min} - 2\sin\beta}{\sin\beta_{\min} - \sin\beta_{\max}}\right)\left(\begin{array}{l}\sin m\lambda,\ m > 0 \\ \cos m\lambda,\ m \le 0\end{array}\right),\ n \ge 0. \quad (9)$$

This is a complete and orthogonal set of functions in the domain $0 \le \lambda \le 2\pi$, $\beta_{\min} \le \beta \le \beta_{\max}$ (Bien et al. [1978]), where P_n represents the Legendre polynomials of order n.

We build the following condition equations for minor planet r and each time t_i^r,

$$\Delta\lambda_i^r = (\lambda_i^r)^{obs} - \lambda_i^r(\overrightarrow{\sigma_r^\circ} + \overrightarrow{\Delta\sigma_r^\circ}; t_i^r) + \sum_{n,m}\Delta l_{n,m}K_{n,m}(\lambda_i^r, \beta_i^r), \quad (10)$$

$$\Delta\beta_i^r = (\beta_i^r)^{obs} - \beta_i^r(\overrightarrow{\sigma_r^\circ} + \overrightarrow{\Delta\sigma_r^\circ}; t_i^r) + \sum_{n,m}\Delta b_{n,m}K_{n,m}(\lambda_i^r, \beta_i^r),$$

where $K_{n,m}$ are the functions (9) and $\Delta l_{n,m}, \Delta b_{n,m}$ are coefficients to be determined. Denoting $(\lambda_i^r)^{obs}$ and $(\beta_i^r)^{obs}$ as the observed values for ecliptic longitude and latitude and performing the square angular residual we obtain,

$$(R_i^r)^2 = (\Delta\lambda_i^r)^2 \cos^2 \beta_i^r + (\Delta\beta_i^r)^2. \tag{11}$$

Adding (11) for all the times and minor planets, the residual function is built

$$R^2(\overrightarrow{\Delta\sigma_1^\circ}, ..., \overrightarrow{\Delta\sigma_N^\circ}; \overrightarrow{\Delta l}, \overrightarrow{\Delta b}) = \sum_{r=1}^{N_a}\sum_{i=1}^{n_r}(R_i^r)^2, \tag{12}$$

where each minor planet has n_r observations and the number of asteroids is N_a and being $\overrightarrow{\Delta l}$, $\overrightarrow{\Delta b}$ vectors of the $\Delta l_{n,m}$, $\Delta b_{n,m}$ coefficients which are ordered following Brosche (1966).

Obviously, we must use a truncated series for the residual function (12) and so, if n ranges from 0 to N and m ranges from $-N$ to $+N$, there are $(N+1)(N+1)$ coefficients to determine. The minimum condition applied to $R^2(\overrightarrow{\Delta\sigma_1^\circ}, ..., \overrightarrow{\Delta\sigma_{N_a}^\circ}; \overrightarrow{\Delta l}, \overrightarrow{\Delta b})$, provides the normal system

$$\begin{bmatrix} S_1 & 0 & . & 0 & T_1^1 & T_1^2 \\ 0 & . & . & 0 & . & . \\ . & . & . & 0 & . & . \\ . & . & 0 & S_{N_a} & T_{N_a}^1 & T_{N_a}^2 \\ (T_1^1)^t & . & . & (T_{N_a}^1)^t & F^1 & 0 \\ (T_1^2)^t & . & . & (T_{N_a}^2)^t & 0 & F^2 \end{bmatrix} \begin{bmatrix} \overrightarrow{\Delta\sigma_1^\circ} \\ \\ \\ \overrightarrow{\Delta\sigma_{N_a}^\circ} \\ \overrightarrow{\Delta l} \\ \overrightarrow{\Delta b} \end{bmatrix} = \begin{bmatrix} \overrightarrow{W_1} \\ \\ \\ \overrightarrow{W_{N_a}} \\ \overrightarrow{U_1} \\ \overrightarrow{U_2} \end{bmatrix}. \tag{13}$$

The quantities in (13) are obviously different from these from the previous section (8), as it is shown in Appendix B.

4. ROTATIONS INDUCED BY THE ANALYTICAL METHOD

In this section our aim is to develop expressions that will enable us to compare the parameters obtained by means of the geometrical method with the analogous parameters induced by the analytical method.

The expression (4) given in § 2 in heliocentrical ecliptic coordinates are,

$$\begin{aligned} \Delta\xi' \sin\lambda\cos\beta + \Delta\eta' \sin\beta &= \Delta\lambda \sin\lambda\cos\beta + \Delta\beta\cos\lambda\sin\beta, \\ -\Delta\xi' \cos\lambda\cos\beta + \Delta\varepsilon' \sin\beta &= -\Delta\lambda\cos\lambda\cos\beta + \Delta\beta\sin\lambda\sin\beta, \\ \Delta\eta' \cos\lambda\cos\beta + \Delta\varepsilon' \sin\beta &= \Delta\beta\cos\beta, \end{aligned} \tag{14}$$

and the infinitesimal rotation matrix (2) is now given by $R(\Delta\xi', \Delta\eta', \Delta\varepsilon')$. Then, for each position (λ, β) we can obtain the numerical values of the expressions on the right hand of these equations, applying the analytical expressions (11). So, if we denote by $f_1(t_i^r)$, $f_2(t_i^r)$, $f_3(t_i^r)$ the equations of condition applied to the position (λ_i^r, β_i^r) belonging to the minor planet r at the time t_i^r, we can build the residual function

$$F^2(\Delta\xi', \Delta\eta', \Delta\varepsilon') = \sum_{r=1}^{N_a}\sum_{i=1}^{n_r}\left[[f_1(t_i^r)]^2 + [f_2(t_i^r)]^2 + [f_3(t_i^r)]^2\right]. \tag{15}$$

The minimum condition applied to (15) provides the normal system,

$$\begin{bmatrix} A & B & C \\ B & E & F \\ C & F & G \end{bmatrix} \begin{bmatrix} \Delta\xi' \\ \Delta\eta' \\ \Delta\varepsilon' \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \tag{16}$$

and the following coefficients are obtained

TABLE 1

COMPARISON OF THE M.A.Q.R. FOR EACH ASTEROID AFTER THE APPLICATION OF THE GEOMETRICAL AND THE ANALYTICAL CORRECTION

		Geometrical		Analytical	
Num.	Name	Num. Obs.	m.a.q.r.	Num. Obs.	m.a.q.r.
1	Ceres	3804	$1''.84$	3010	$1''.69$
3	Palas	3389	$1''.59$	2343	$1''.40$
4	Vesta	3111	$1''.76$	2421	$1''.50$
6	Hebe	2179	$1''.71$	999	$1''.54$
7	Iris	1804	$1''.77$	1768	$1''.57$
11	Partenope	1789	$1''.66$	1765	$1''.48$
18	Melpomene	1596	$1''.63$	1368	$1''.43$
39	Laetitia	2305	$1''.67$	1828	$1''.51$
40	Harmonia	1481	$1''.84$	1473	$1''.71$

$$A = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} (cos^2\beta i^r), \quad B = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \tfrac{1}{2} \sin\lambda_i^r \sin 2\beta i^r, \quad C = -\sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \tfrac{1}{2} \cos\lambda_i^r \sin 2\beta_i^r,$$

$$E = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} (1 - \sin^2\lambda_i^r \cos^2\beta_i^r), \quad F = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \sin 2\lambda_i^r \cos^2\beta_i^r,$$

$$G = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \left[1 - \cos^2\lambda_i^r \cos^2\beta_i^r\right], \quad D_1 = -\sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \Delta\lambda_i^r \cos^2\beta_i^r, \tag{17}$$

$$D_2 = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \left[\frac{1}{2}\Delta\lambda_i^r \sin\lambda_i^r \sin 2\beta_i^r + \Delta\beta_i^r \cos\beta_i^r\right],$$

$$D_3 = \sum_{r=1}^{N_a} \sum_{i=1}^{n_r} \left[-\Delta\lambda_i^r \cos\lambda_i^r \sin 2\beta_i^r + \Delta\beta_i^r \sin\beta_i^r\right].$$

Finally, the incremental values $\Delta\xi', \Delta\eta', \Delta\varepsilon'$ obtained must be related with the values $\Delta\xi, \Delta\eta, \Delta\varepsilon$ that refer to the equatorial heliocentric system.

5. NUMERICAL RESULTS AND COMMENTS

To obtain numerical results we have selected nine minor planets whose orbits lie in a band around the Ecliptic ranging from $\beta_{\min}$ =-15° to $\beta_{\max}$ = +15°. We have integrated the Planetary Lagrange equations backwards from JD2451545.0 to JD2417566.47784 taking the initial elements from (Batrakov 1997) and making use of VSOP87 planetary theory (Bretagnon & Francou [1988]). In Table 1 we list the number of observations employed after their individual corrections, their mean angular quadratic residual (m.a.q.r.) values and the same data after the analytical correction with $N = 4$.

A difficult problem arises in selecting the maximum degree N in the analytical development (10). We found that the method proposed by Bien (1978) is not suitable in this case because of the large number of observations involved in the adjustments. For this reason, we have used an alternative method taking into account the covariance function which measures the statistics correlation between two corrections for each spherical coordinates among the asteroids. In our case, the covariance between the corrections for a given point a and another a' is given by Heiskanen & Moritz (1985),

$$C(d) = \overline{\Delta a \Delta a'},$$

being d the angular distance between both positions. In Figure 1 and Figure 2 we see the functions of covariance for the adjustments in Ecliptic longitude and latitude for $N = 2, 4$ also, we have included the covariance a priori, which is clearly improved by the adjustments. The oscillations around the x-axis may be considered as zero to the precision used. We have omitted $N = 3$ in these figures because the lines representing $N = 3$ and $N = 4$ seem to coincide due to the scale employed on the figures. In Figure 3 and Figure 4 we have magnified the left part of the corresponding figures for $N = 3$ and $N = 4$ in order to see the differences between them at the origin, again the covariance a priori is included. From these figures, we see that the adjustment for $N = 4$ improves slightly the adjustment for $N = 3$, so $N = 4$ is the value taken.

The parameters (16) obtained from the method presented in § 4 are

$$[\Delta\xi', \Delta\eta', \Delta\varepsilon'] = [-0''.064, -0''.008, -0''.016], \tag{18}$$

which provide the equatorial parameters

$$[\Delta\xi, \Delta\eta, \Delta\varepsilon] = [-0''.056, -0''.033, 0''.016]. \tag{19}$$

From the geometrical method exposed in § 2 the following parameters for (2) are obtained

$$[\Delta\xi, \Delta\eta, \Delta\varepsilon] = [-0''.041, -0''.009, 0''.015]. \tag{20}$$

The comparison of (20) and (19) shows very similar values for $\Delta\xi$ and $\Delta\varepsilon$.

6. CONCLUSIONS

Of the two methods presented, the geometrical one is useful to study the differences between the catalogue references, materialized through the infinitesimal angles round the axes. The analytical method is more convenient to the study of the errors depending on the zones of the celestial sphere. In the present case, the analytical method has been applied to the study of a band of 15° around the ecliptic, but the method may be modified to be applied in any domain of the celestial sphere, even if the amplitude of the domain is not constant.

To validate the geometrical method the corrections induced to the Vernal point were studied and it was found a great coincidence between the values obtained and the ones proposed by Batrakov (1999). The analytical method may not be validated directly, instead we compared it with the geometrical method and we found that the geometrical corrections induced by the analytical methods were in great accordance with those obtained through the geometrical method.

APPENDIX A

PERTURBATION ELEMENTS TECHNIQUE

Let (a, b) be the spherical, equatorial or ecliptic, coordinates and we compute the incremental values

$$a(\sigma_i^\circ + \Delta\sigma_i^\circ; t) = a(\sigma_i^\circ; t) + \sum_{m=1}^{6} \left.\frac{\partial a}{\partial \sigma_m^\circ}\right|_t \Delta\sigma_m^\circ, \tag{21}$$

where we have

$$\left.\frac{\partial a}{\partial \sigma_m^\circ}\right|_t = \sum_{k=1}^{6} \frac{\partial a}{\partial \sigma_k} \left.\frac{\partial \sigma_k}{\partial \sigma_m^\circ}\right|_t, \tag{22}$$

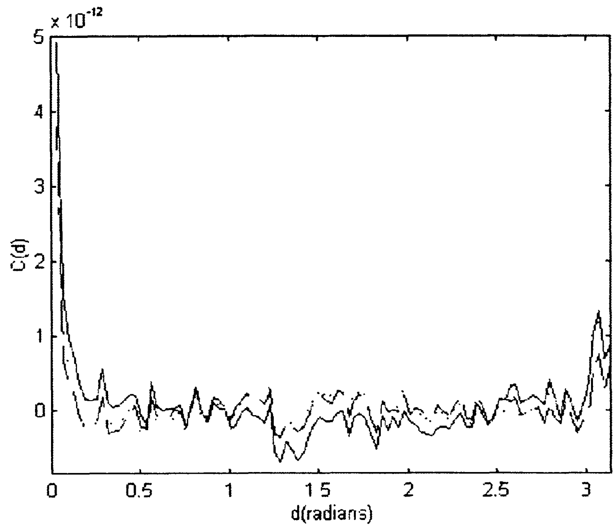

Fig. 1. Behavior of the covariance functions for the Ecliptic longitude. The solid line is used for the covariance a priori, the dotted line is used for $N = 2$ and the dashed line for $N = 4$.

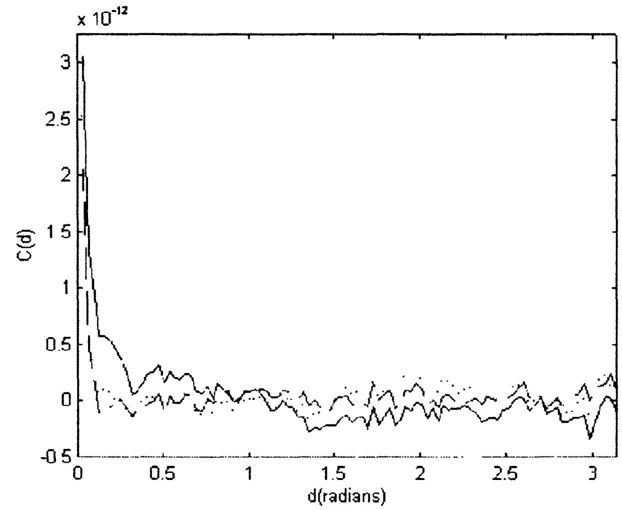

Fig. 2. Behavior of the covariance functions for the Ecliptic latitude. The solid line is used for the covariace a priori, the dotted line is used for $N = 2$ and the dashed line for $N = 4$.

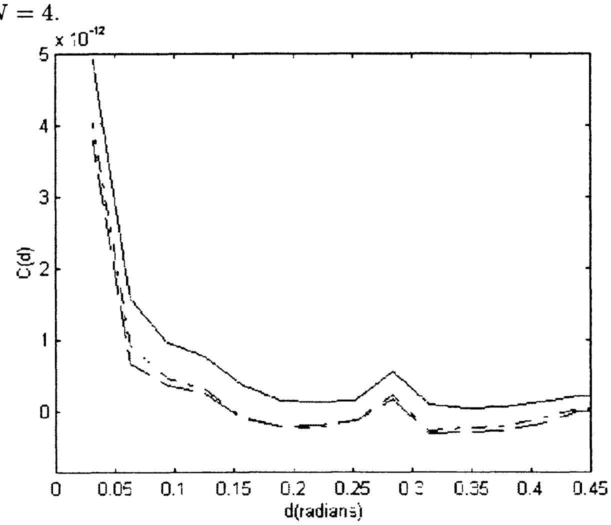

Fig. 3. Detail of the covariance functions for the Ecliptic longitude. The solid line is used for the covariance a priori, the short dashed line is used for $N = 3$ and the long dashed line for $N = 4$.

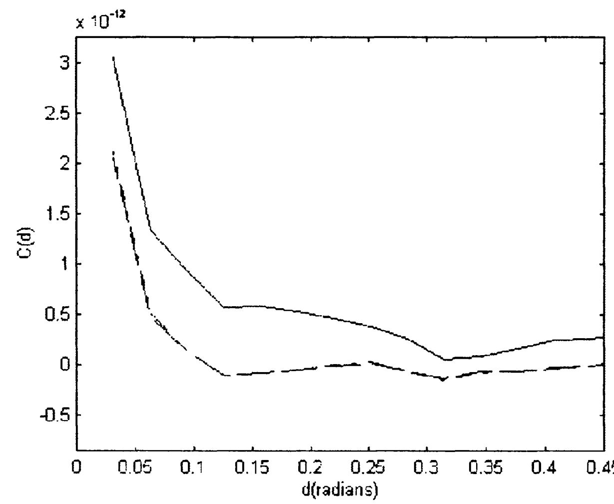

Fig. 4. Detail of the covariance functions for the Ecliptic latitude. The solid line is used for the covariance a priori, the short dashed line is used for $N = 3$ and the long dashed line for $N = 4$.

and we denote by $\sigma^{\circ}_{k,m}(t) = \frac{\partial \sigma_k}{\partial \sigma^{\circ}_m}(t)$ this functional dependence of the perturbed elements. The computation of these partial derivatives is made by means of the resolution of a system of differential equations given by the planetary Lagrange equations and their partial derivatives with respect to the initial elements

$$\frac{d}{dt}\sigma_j = \sum_{k=1}^{6} L_{j,k}\frac{\partial \Re}{\partial \sigma_k}, \tag{23}$$

$$\frac{d}{dt}\left\{\frac{\partial \sigma_k}{\partial \sigma^{\circ}_j}\right\} = \sum_{i=1}^{6}\sum_{m=1}^{6}\left\{\frac{\partial L_{k,m}}{\partial \sigma_i}\frac{\partial \Re}{\partial \sigma_m} + L_{k,m}\frac{\partial^2 \Re}{\partial \sigma_i \partial \sigma_m}\right\}\frac{\partial \sigma_i}{\partial \sigma^{\circ}_j}$$

where L is the Lagrange matrix, $\Re$ is the perturbation function and all the considered indexes range from 1 to 6.

APPENDIX B

NORMAL SYSTEM COEFFICIENTS

For any spherical coordinates (a, b), we have the residuals,

$$\Delta a^r_i = (a^r_i)^{obs} - a(\overrightarrow{\sigma^{\circ}_r} + \overrightarrow{\Delta\sigma^{\circ}_r}; t^r_i) + \left[\overrightarrow{\Gamma_a}(a^r_i, b^r_i)\right]^t . \overrightarrow{\Delta\phi}, \tag{24}$$

$$\Delta b^r_i = (b^r_i)^{obs} - b(\overrightarrow{\sigma^{\circ}_r} + \overrightarrow{\Delta\sigma^{\circ}_r}; t^r_i) + \left[\overrightarrow{\Gamma_b}(a^r_i, b^r_i)\right]^t . \overrightarrow{\Delta\phi},$$

where $\overrightarrow{\Gamma}$ and $\overrightarrow{\Delta\phi}$ depend (in a linear approach) on the different adjustment models. Also, from Appendix A,

$$\begin{aligned} A_m(t) &= \left.\frac{\partial a}{\partial \sigma^{\circ}_m}\right|_t, \text{and } D_m(t) = \left.\frac{\partial b}{\partial \sigma^{\circ}_m}\right|_t, \\ \overrightarrow{A}(t) &= (A_m(t))^t, \text{and } \overrightarrow{D}(t) = (D_m(t))^t, \end{aligned} \tag{25}$$

to obtain

$$\Delta a^r_i = (B^r_i) - \left[\overrightarrow{\Delta\sigma^{\circ}}\right]^t . \overrightarrow{A}(t^r_i) + . \left[\overrightarrow{\Delta\phi}\right]^t . \overrightarrow{\Gamma_a}(a^r_i, b^r_i), \tag{26}$$

$$\Delta b^r_i = (C^r_i) - \left[\overrightarrow{\Delta\sigma^{\circ}}\right]^t . \overrightarrow{D}(t^r_i) + \left[\overrightarrow{\Delta\phi}\right]^t . \overrightarrow{\Gamma_b}(a^r_i, b^r_i),$$

where $B^r_i = (a^r_i)^{obs} - a(\overrightarrow{\sigma^{\circ}_r}; t^r_i)$ and $C^r_i = (b^r_i)^{obs} - b(\overrightarrow{\sigma^{\circ}_r}; t^r_i)$. Denoting the product $\overrightarrow{p} \otimes \overrightarrow{q} = \overrightarrow{p}.\overrightarrow{q}^t$ being $\overrightarrow{p}$ and $\overrightarrow{q}$ two vectors, we have,

$$\begin{aligned} S_r &= \sum_{i=1}^{n_r}\left[\overrightarrow{A} \otimes \overrightarrow{A}\cos^2 b + \overrightarrow{D} \otimes \overrightarrow{D}\right]^r_i, \\ T_r &= \sum_{i=1}^{n_r}\left[\overrightarrow{\Gamma_a} \otimes \overrightarrow{A}\cos^2 b + \overrightarrow{\Gamma_b} \otimes \overrightarrow{D}\right]^r_i, \end{aligned} \tag{27}$$

$$W_r = \sum_{i=1}^{n_r}\left[\overrightarrow{A}.B\cos^2 b + \overrightarrow{D}.C\right]^r_i,$$

$$F = \sum_{r=1}^{N_a}\sum_{i=1}^{n_r}\left[\overrightarrow{\Gamma_a} \otimes \overrightarrow{\Gamma_a}\cos^2 b + \overrightarrow{\Gamma_b} \otimes \overrightarrow{\Gamma_b}\right]^r_i,$$

$$\overrightarrow{U_1} = -\sum_{r=1}^{N_a}\sum_{i=1}^{n_r}\left[\overrightarrow{\Gamma_a}.B\cos^2 b + \overrightarrow{\Gamma_b} \otimes C\right]^r_i .$$

For the analytical approach we can put $\overrightarrow{T_r} = \left[T^1_r T^2_r\right]^t$ and $\overrightarrow{\Delta\phi} = \left[\overrightarrow{\Delta\xi}^t \overrightarrow{\Delta\eta}^t\right]^t$ and all the formulas are identical to the geometrical ones, having $\overrightarrow{[U_1, U_2]}$ instead of $\overrightarrow{U_1}$.

REFERENCES

Batrakov Y. V. 1997. Ephemerides of Minor Planets for 1998 (St. Pettersbourg, Russia: Inst. Applied Astronomy)
Batrakov Y. V., Chernetenko, Y. A., Gorel, G. K., & Gudkova, L. A. 1999. A&A, 352, 703
Bien, R., Fricke, W., Lederle, T., & Schwann, H. 1978, Veröffentlichungen Astron. Rechen Institut. Heilderberg. Nr 29
Bretagnon, P., & Francou, G. 1988. A&A, 202, 309
Brosche, P. 1966, Veröffentlichungen Astron. RechenInstitut Heidelberg, Nr. 17
Duma, D. P. 1991, Proceedings of the 1st. Spain-URSS Workshop on Positional Astronomy and Celestial Mechanics, eds. A. López García, R. López Machí, & A. G. Sokolsky (Valencia: University of Valencia Press), 139
Heiskanen, W., & Moritz, H. 1985. Geodesia Física (Madrid: Instituto geográfico nacional. Instituto de Astronomía y Geodesia)
Levallois, J. J. 1969, Géodésie générale, (Éditions Eyrolles)
Marco, F. J., López, J. A., & Martínez, M. J. 1996, in IAU Symp. 172, Dynamics, Ephemerides and Astrometry of the Solar System, eds. S. Ferraz-Mello, B. Morando, & J.-E. Arlot (The Netherlans: Kluwer), 199
________. Cel. Mech., 68, 193
Pierce, D. A. 1971, AJ, 76, 177

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