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ORBIT OF COMET 122P/ DE VICO

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RESUMEN

Se calcula una órbita para el Cometa 122P/ de Vico. La órbita se basa en 400 observaciones en ascensión recta y 396 en declinación hechas entre el 20 de febrero de 1846 y el 16 de mayo de 1996, pero con un intervalo de 150 años entre las observaciones del siglo XIX y las del siglo XX. Fue difícil enlazar la órbita de 1846 con la de 1995 y fue necesario un enfoque de condiciones del contorno para enlazar ambas órbitas en un punto común. El sistema lineal que resulta tiene forma de matriz de banda simétrica y negativa definida, que se resuelve fácilmente con la descomposición de Cholesky libre de raíz cuadrada. Pruebas estadísticas indican que la órbita es buena.

ABSTRACT

An orbit for Comet 122P/ de Vico, based on 400 observations in right ascension and 396 in declination, made between 20 February 1846 and 16 May 1996 but with a gap of 150 years between the 19th and the 20th century observations, is given. Linking to two orbits proved difficult until a boundary value approach was taken: link best fit orbits for the comet to a common fitting point. The linear system to be solved acquires a special form, symmetric, negative-definite, band, that permits efficient solution by square-root free Cholesky decomposition. Statistical tests indicate that the orbit is good.

Key Words: **CELESTIAL MECHANICS — COMETS: INDIVIDUAL (COMET 122P/ DE VICO) — METHODS: DATA ANALYSIS**

1. INTRODUCTION

Comet 122P/ de Vico is included in the *Catalogue of Cometary Orbits 2003* (Marsden & Williams 2003). Why, therefore, devote effort to a study of its orbit? There are two reasons. First, the catalogued orbit is based on 199 observations, but a literature search shows that double that number is available. To include *all* of the observations, and in particular the 19th century observations, in the orbit seems a worthwhile endeavor. As Dubyago (1961) sagely remarks, "...the concept of a definitive orbit is relative...the possibility often exists for further improvement of the definitive orbit in connection with a new discussion and reduction of the observations...." The second reason for calculating the orbit arises because the endeavor entails certain challenges.

Comet 122P/ de Vico was discovered on 20 February 1846 by Father de Vico in Rome. 1846 seems to have been an *annus mirabilis* for de Vico, who discovered four comets that year, sharing credit for one of the discoveries with Hind. Of these four 122P/ de Vico, which I shall hereafter refer to as simply de Vico, is the most interesting. The orbit calculated from the first observations was similar to that of another comet discovered that year, 5D/ 1846 D2 (Brorsen), so similar that the two were considered by some to be the same. In the *Astronomische Nachrichten* for 1846 (Vol. 23, Nr. 556, p. 62) we read (translated from German): "There is thus no doubt that the comet discovered by Brorsen on 8 March is the one de Vico discovered on 20 February." The observers at Altona, Germany, published observations of Comet de Vico as observations of

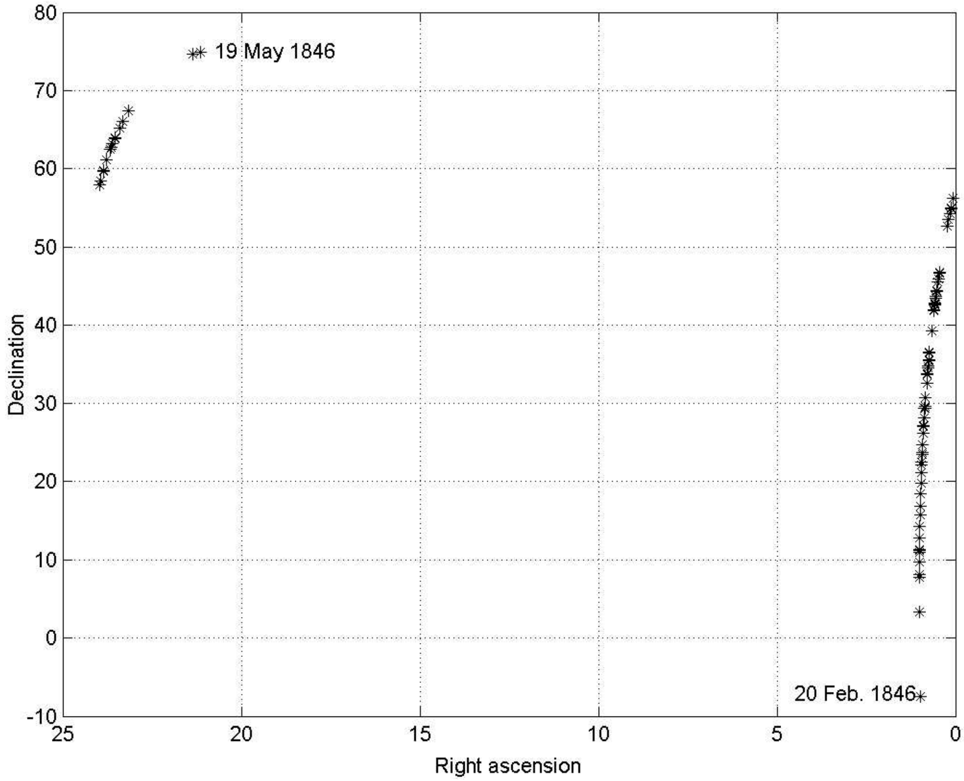

Fig. 1. The 1846 observations.

Brorsen's comet (*Astronomische Nachrichten*, 1846, Vol. 23, Nr. 556, pp. 61-62). More refined calculations, however, showed that the two are different. Thus, the *Monthly Notices RAS* for 1846 (Vol. 7, p. 79) mentions, "this comet has so much similarity with that of Brorsen, that at first the two were suspected to be identical". The calculations showed that Brorsen's comet has a period of 5.6 years whereas de Vico's period is ≈ 75 years. De Vico's comet was observed until May of 1846 and then not seen again until 1995, when it was observed between September 1995 and May 1996. It should have made an appearance in 1922, but was apparently not observed that year. Seeing as even a few observations from 1922 would greatly strengthen the solution, it is imperative to ascertain if there were in fact no observations made.

One must link the relatively poor 1846 visual observations made during four months with the more precise 1995–1996 CCD observations. This proved a real challenge, and the solution I found, treat linking the two orbits as a boundary value problem, may prove useful to others who face a similar situation. The problem becomes one of determining a best fit orbit to both groups. The best fit orbit for the 1846 observations represents poorly the 20th century observations and the same happens when the best fit orbit for those observations is extended back to 1846. So poorly that differential corrections to improve both orbits simultaneously diverge. But if we use boundary value techniques the problem becomes tractable: solve for an intermediate fitting point that satisfies both groups of observations and then employ this fitting point as a good first approximations to the differential corrections.

2. PRELIMINARY DATA REDUCTION AND EPHEMERIDES

I conducted a literature search of the journals published in the 19th century that include comet observations and also annual reports of some of the major observatories. Observations of Comet de Vico for 1846 were found in *The Astronomical Journal, Monthly Notices RAS,* and *Astronomische Nachrichten.* This yielded a total of 137 observations in right ascension (α) and 133 in declination (δ). I have discussed the reduction of observations in a previous publication (Branham 2003) and will only repeat certain salient features. Observations were

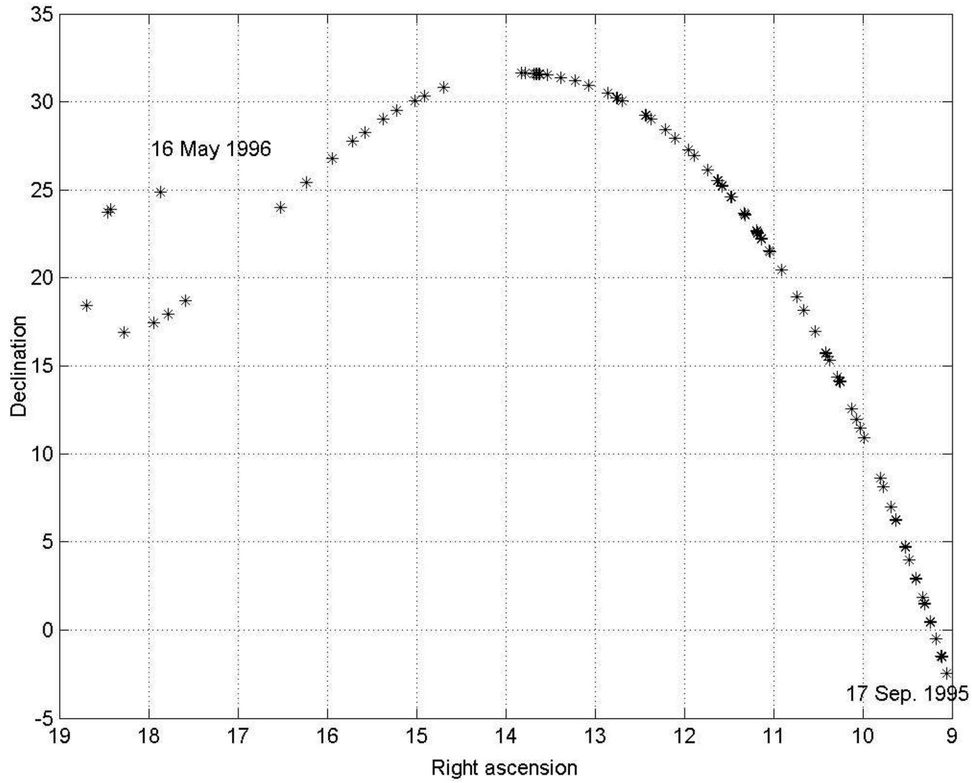

Fig. 2. The 1995–1996 observations.

TABLE 1

OBSERVATORIES

Observatory	Obsns. in α	Obsns. in δ	Reference[a]
Vienna, Austria	4	3	AN, 1846, Vol. 23, pp. 193-94
Cambridge, England	3	3	AN, 1846, Vol. 23, pp. 69-70
Altona, Germany	30	28	AN, 1846, Vol. 23, pp. 44-46, 61
Berlin, Germany	8	8	AN, 1848, Vol. 26, p. 3
Bonn, Germany	24	24	AN, 1846, Vol. 23, pp. 257-58, 295-96
Hamburg, Germany	11	11	AN, 1846, Vol.23, pp.77 – 78 MN, 1846, Vol.7, p.87, 91
Mailand, Germany	1	1	AN, 1846, Vol. 23, p. 62
Padua, Italy	9	9	AN, 1846, Vol. 23, pp.189-90, 275-76
Rome, Italy	1	1	MN, 1846, Vol. 7, p. 79
Leiden, Netherlands	2	2	AN, 1846, Vol. 23, pp. 201-02
Cambridge, USA	25	24	AN, 1846, Vol.23, pp.91 – 92 MN, 1846, Vol.7, p.187
(Old) U.S. Naval, USA	19	19	AJ, 1850, Vol. 1, p. 137
Total	137	133	

[a] AJ:*Astronomical Journal*; AN: *Astronomische Nachrichten*; MN: *Monthly Notices RAS.*

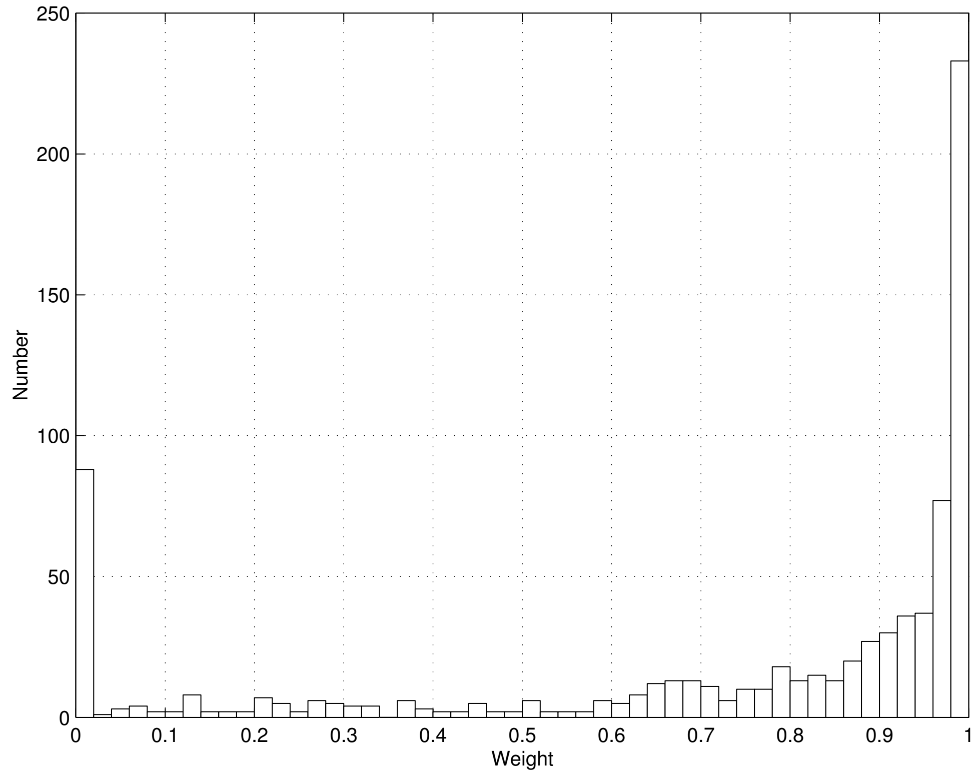

Fig. 3. Distribution of the weights.

reduced to the common format of: Julian Ephemeris Day (JED), right ascension, and declination. (I use the term "Ephemeris Day" realizing that ephemeris time has been replaced by Terrestrial Time. But for the year 1846 the two systems are the same, and it is more precise, albeit perhaps slightly arcane, to use the older designation of ephemeris time.) Such a reduction was necessary because some observers preferred to measure α in circular rather than time units, some preferred north polar distance rather than δ, some used sidereal rather than mean time, and most, with the exception of the British observers, used time of place rather than Greenwich Mean Time to record the observation.

The observations were all made with equatorial telescopes and filar or ring micrometers, the comet measured with respect to a nearby reference star. Given that modern star catalogues are more precise than 19th century catalogues, it is more accurate when a definite reference star is mentioned to recalculate its apparent position, using the algorithm in Kaplan et al. (1989), from a recent modern catalogue, Tycho-2 (Høg et al. 2000), and apply $\Delta\alpha$ and $\Delta\delta$, corrected for differential aberration, to the new position. If $\Delta\alpha$ and $\Delta\delta$ were not given the differences in the positions between the older catalogue and Tycho-2 were applied to the published positions of the comet. If no reference star was given, then one had to take the observation as published.

The 20th century observations, found on various IAU and Minor Planet Center circulars, seem all to have been made by CCD's and totalled 263 in each coordinate; there were no α only or δ only observations. (In case one wonders how a CCD observation can be one coordinate only, the answer is simple. CCD's are used on modern meridian instruments. A problem with the clock results in a δ only observation: a circle error produces an α only observation.) The format for all of the observations was standard: universal time, α, and δ referred to equinox J2000. Between the 19th and the 20th century observations there was thus a total of 400 observations in α and 396 in δ. Figure 1 graphs the 1846 observations and Figure 2 the 1995–1996 observations. Because the 20th century observations can be found with relative facility whereas the 19th century observations are scattered over the literature, I have included in Table 1 a summary of the 19th century observations.

The rectangular coordinates and velocities of the comet and the Earth were calculated by a program, used in numerous investigations previously, that treats the solar system as an n-body problem. The program is a

TABLE 2

ERRORS AND MISSING INFORMATION IN THE OBSERVATIONS

Reference	Date	Error or Missing Data
AN, 1846, Vol. 23, pp. 61-61	All	Obsns. for Comet de Vico, not Brorsen
AN, 1846, Vol. 23, p. 189	March 1	α probably $01^{\rm h}00^{\rm m}34.^{\rm s}8$
AN, 1846, Vol. 23, p. 189	March 7	δ probably $+16°46'33''$
AN, 1846, Vol. 23, pp. 193-94	April 4	Unidentified star is Tycho 9537-00387-1
AN, 1846, Vol. 23, pp. 201-02	March 23	Unidentified star is Tycho 2288-00102-1
AN, 1846, Vol. 23, pp. 201-02	March 31	Unidentified star is Tycho 9537-00387-1
AN, 1848, Vol. 26, p. 3	March 30	Unidentified star is Tycho 2805-01621-1
AN, 1848, Vol. 26, p. 3	April 6	δ probably $03°00'54.''5$
MN, 1846, Vol. 7, p. 187	May 4	δ probably $67°21'12''$

12-th order Lagrangian predictor-corrector that incorporates relativity by a Schwarzschild harmonic metric. To obtain coordinates and velocities for the Earth, the Moon is carried as a separate body. This means a small step-size, $0.^{d}25$. To correct the comet's orbit partial derivatives are calculated by Moulton's method (Herget 1968), which integrates the partial derivatives to correct for the osculating rectangular coordinates and velocities along with the coordinates and velocities. The rectangular coordinates, after interpolation to the moment of observation for the Earth and to the moment of observation antedated by the light time correction to allow for planetary aberration, are then converted to a unit vector that is transformed to a mean or apparent place in α and δ by application of precession, nutation, annual aberration, relativity, and so forth. The 19th century, but not the 20th century, observations were corrected for the E-terms of the aberration during the calculation of a mean place. The final step calculates an observed minus a computed place, (O-C), in α and δ. To interpolate coordinates and partial derivatives I used recursive Aitken interpolation (Branham 2003).

3. ERRORS OR MISSING INFORMATION IN THE 1846 OBSERVATIONS

When looking at large (O-C)'s it sometimes becomes evident that there is some sort of error with the published observation. When the error is corrected, the (O-C) becomes acceptable. Because all of the errors were associated with the 1846 observations, I will restrict my comments to these. Typical errors include misidentifying a reference star. The wrong reference star can sometimes be detected by taking an abnormally large (O-C) and seeing if there is a relatively bright star near the published reference star that would reduce the (O-C) to something reasonable. Sometimes a differential observation gives the wrong sign for a $\Delta\alpha$ or $\Delta\delta$. Simply changing the sign produces a good (O-C). And sometime a clerical error occurs, such as writing a 3 for a 5, a 5 for 8, or a 1 for a 7. Typesetting was done in those days from a written manuscript where it is easy to confuse these numbers. If, for example, a declination is published ending with something like $54.''6$ and the (O-C) is close to $20''$, it is reasonable to assume that the declination really should have been $34.''6$. The assumption of a clerical error can be checked by looking at the daily motion of the comet to see if the time of observation indicates that a change in the coordinate becomes likely.

The Altona, Germany, observers made a series of observations closely spaced in sidereal time and then averaged the time, α, and δ and published the result as one observation in mean time. I used the original, unaveraged sidereal time observations.

Table 2 lists the errors found for the benefit of anyone who wishes to further study and perhaps improve the orbit of this comet.

4. TREATMENT OF THE OBSERVATIONS

Given the disparity in the quality of the observations, one should contemplate ways of assigning higher weight to the better observations. Differential observations referred to an apparent place calculated from

Tycho-2 should be better than apparent places published by the observer, but this is not invariably so. I used the same weighting scheme as in my previous publication (Branham 2003), the biweight. One scales the post-fit residual r_i by the median of the residuals and assigns a weight wt as

$$\begin{aligned} wt &= [1-(r_i/4.685)^2]^2; r_i \leq 4.685 \\ wt &= 0; r_i \succ 4.685\,. \end{aligned} \tag{1}$$

The robust L_1 criterion (Branham 1990a) calculates the first approximation. Because the first approximation is good, it becomes unnecessary to iterate the solutions. The median weight was 0.91. Figure 3 shows the distribution of the weights. 72.9% of the observations received weights between 0.7 and 1, 67.1% weights between 0.8 and 1, 52.6% weights between 0.9 and 1. Fifty-four observations received weight of less than 0.1, of which six were 0.

Linking the 1846 with the 1995–1996 observations proved difficult. At first I calculated a best fit orbit through the 20th century observations for epoch JD 2450000.5 and then extended it back to 1846. But this resulted in high residuals. Apparently the time span is too short to give a good fit to the 19th century observations. Then I reversed the procedure: determine a best fit orbit for the 19th century observations using epoch JD 2395310.5, with a four month time span, and extend it to 1995. But once again the residuals were unacceptably high, sufficiently high to destroy convergence of the differential corrections. Thus, it proved impossible to correct either the two best fit orbits to encompass all of the observations.

The convergence of differential corrections, of course, is a complicated problem. The fact that an orbit for Comet 122P/ de Vico has already been published in Marsden & Williams's catalogue (2003) shows that perhaps the previous orbit computer did not run into the difficulties that I did. My problems may have been caused or exacerbated by carrying the Moon as a separate body. (My n-body program integrates both coordinates and the partial derivatives needed for the differential corrections.) The Moon's motion is complicated and others, such as Oesterwinter and Cohen (1972), have found that n-body integrations that include the Moon are much more demanding than merely integrating planetary orbits: among other factors one must use a small step-size, which I took as $0.^d25$. Whether or not this was the cause, I had difficulty in linking the two orbits.

But one can circumvent the difficulty of linking the two sets of observations if one converts the problem to a boundary value problem and enforces the condition that the two orbits have to agree at an intermediate fitting point. We could, for example, enforce the condition that the best fit 1846 orbit and the best fit 1995–1996 orbit calculate the same rectangular coordinates and velocities for JD 2420000.5.

5. THE RELAXATION METHOD

The equations of perturbed motion in celestial mechanics, without inclusion of relativistic terms, are

$$\ddot{x} = -Gx(1+m)/r^3 + \sum_{j=1}^{p} Gm_j[(x_j - x)/\rho_j^3 - x_j/r_j^3]\,, \tag{2}$$

with similar equations in y and z. In Eq. (2) x represents one of the rectangular coordinates of the object whose motion we are studying, m its mass, which for comets may be taken as zero, r its heliocentric distance, x_j the rectangular coordinate of a perturbing planet with heliocentric distance r_j, ρ_j the distance of the comet from the perturbing planet, p the number of perturbing planets, and G is the Gaussian gravitational constant of 0.01720209895^2. Relativistic terms are not included in Eq. (2) because a finite difference approximation to the equations yields a formal error higher than the error represented by omission of those terms and their inclusion complicates the matrix that ensues from the approximation because it becomes unsymmetric.

To achieve sufficient precision in the approximation the finite differences should be of high order. Ferziger (1981) gives equations for a fourth order method that results in a almost pentadiagonal matrix. Because the matrix is negative definite it can be solved by LU decomposition without pivoting. But it is questionable whether fourth-order is sufficient for numerical integrations in celestial mechanics. I did some experimentation with the fourth-order method, even developing code for the solution of the band matrix, but found out that in the end the solution was of insufficient precision. Then I decided to implement a tenth-order method, an order that should incorporate sufficient precision. This decision proved felicitous because the resulting matrix exhibits pleasing features. The matrix becomes a strict band matrix, a matrix whose nonzero elements are constrained to lie on diagonals immediately above and below the main diagonal. The number of upper diagonals is called the "upper bandwidth" and the number of lower diagonals the "lower bandwidth". The matrix is also symmetric and negative definite and can, therefore, be solved quickly by square-root free Cholesky decomposition. Because Cholesky decomposition requires no pivoting, the bandwidth of the matrix remains constant.

But one must first develop a tenth-order method. A literature search revealed no finite difference approximation of that order. But Cynar (1987) discusses how it can be done using Taylor series truncated after the tenth order. Assume that the function is tabulated at equally spaced intervals h, with h=1, ..., $f_{-k}, f_{-k+1}, \ldots, f_{-1}, f_0, f_1, \ldots, f_{k-1}, f_k, \ldots$. The tenth-order approximation to the second derivative becomes

$$f_0''(x) = \sum_{i=-5}^{-1} c_{i+6} f_i + \sum_{i=1}^{5} c_{i+5} f_i - (\sum_{i=1}^{10} c_i) f_0(x), \tag{3}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \\ c_8 \\ c_9 \\ c_{10} \end{pmatrix} = \begin{pmatrix} 0.00031746031746 \\ -0.00496031746032 \\ 0.03968253968254 \\ -0.23809523809525 \\ 1.66666666666668 \\ 1.66666666666668 \\ -0.23809523809525 \\ 0.03968253968254 \\ -0.00496031746032 \\ 0.00031746031746 \end{pmatrix}.$$

with $\sum_{i=1}^{10} c_i$ =2.92722222222222. Let us now postulate that we have osculating coordinates and velocities for the comet for a certain date in 1846, denoted by $\mathbf{x}_0$ and $\dot{\mathbf{x}}_0$, and for another date in 1995, denoted by $\mathbf{x}_n$ and $\dot{\mathbf{x}}_n$. These coordinates form the boundary conditions. We really do not need the velocities, which can be calculated from the coordinates by numerical differentiation. One establishes a linear system for the $3(n-1)$ unknowns $\mathbf{x}_1, \ldots, \mathbf{x}_{n-1}$. The equation for an interior point k for the x-coordinate assumes the form

$$\sum_{i=1}^{5} c_i x_{k-i} - 2.92722222222222 x_k + \sum_{i=6}^{10} c_i x_{k+i-5} = \frac{-G x_k}{r_k^3} + \sum_{j=1}^{p} G m_j [\frac{(x_{k,j} - x_k)}{\rho_{kj}^3} - \frac{x_{kj}}{r_{kj}^3}]. \tag{4}$$

Similar equations pertain to y and z. The system will be represented by a band matrix. Eq. (4) leads to five superdiagonals and five subdiagonals. But because the coefficients are symmetric, the matrix will be also, and the subdiagonals need not be stored. The bandwidth becomes five. Boundary value problems have difficulties at the boundaries themselves. When $k = 0$ and $k = n$ the values x_0 and x_n are indeed available, but what about $x_{-1}, x_{-2}, \cdots, x_{n+1}, x_{n+2}, \cdots$? In celestial mechanics, fortunately, these values offer no problem because they can be generated by extending the integration to include them. Our matrix, therefore, remains a strict band matrix whereas Ferziger (1981) has to look for a different order approximation near the boundaries to avoid the pre- and post-boundary values; this destroys the strict band structure near the boundaries.

Although Eq. (4) results in a linear matrix for the unknowns, because the unknown x_k appears on the right-hand-side of Eq. (4), the problem itself is nonlinear. One must have or assume a first approximation for the $x's$, use it in the right-hand-side, solve for the $x's$, and iterate until convergence. The matrix is negative definite, as a calculation of its singular values shows. I have previously shown (Branham 1990b) how such a matrix can be stored and solved. To recapitulate briefly, if q is the bandwidth, equal to 5 when we use Eq. (4), the matrix may be stored in column order as a vector occupying $n(q+1) - q(q+1)/2$ storage locations. To locate an i, j matrix element in the vector we use a mapping function that transforms the two indices i and j into a single index k. The mapping function becomes: if $j < q + 1$ then $k = j(j-1)/2 + i$; otherwise $k = qj + i - q(q+1)/2$.

Because the matrix is negative definite, converted trivially to positive definite by multiplication by -1, it may be solved by Cholesky decomposition. But being a band matrix, the square root-free version of the decomposition (Branham 1990a) saves considerable time by obviating the calculation of the numerous square roots. If $\mathbf{A}$ represents the matrix the decomposition is

$$\mathbf{A} = \mathbf{S}^T \cdot \mathbf{D} \cdot \mathbf{S}, \tag{5}$$

where $\mathbf{S}$ is upper triangular with unit principal diagonal and $\mathbf{D}$ is diagonal. For band matrices Eq. (5) becomes:

$$\begin{array}{l} d_i = A_{ii} - \sum_{k=1}^{i-1} d_k S_{ki}^2, i = 1, \ldots, n \\ S_{ij} = (A_{ij} - \sum_{k=1}^{i-1} d_k S_{ki} S_{kj})/d_i, j = i, \ldots, \min(i+q, n)\,. \end{array} \tag{6}$$

To solve the system for the unknown vector $\mathbf{x}$, use an auxiliary vector $\mathbf{y}$. First solve the upper triangular system $\mathbf{S}^T \cdot \mathbf{y} = \mathbf{b}$, where $\mathbf{b}$ represents the right-hand-side, and then the lower triangular system $\mathbf{D} \cdot \mathbf{S} \cdot \mathbf{x} = \mathbf{y}$. The operation count for banded Cholesky decomposition and subsequent calculation of the solution becomes $q^2 n/2 + 7qn/2 + n$.

Banded Cholesky decomposition compares favorably, extremely favorably when $n \gg 1$ and $q \ll n$, the usual situation, with its unbanded counterpart. Cholesky decomposition for a general symmetric matrix requires $n(n+1)/2$ storage locations and has an operation count of $n^3/6 + n^2$. The banded variety, therefore, needs $\approx 2(q+1)/(n+1)$ less memory and $\approx 3q/n$ fewer operations. For Comet de Vico $q = 5$ and, as we shall see, $n = 164040$. Banded decomposition calculates a solution quickly whereas unbanded decomposition exhausts available memory.

But as mentioned previously the procedure is nonlinear and must be iterated. What we are solving is the banded linear system

$$\mathbf{A} \cdot \mathbf{x}_{k+1} = f(\mathbf{x}_k). \tag{7}$$

with first approximation $\mathbf{x}_0$ given by the best fit 1846 and 1995–96 orbits integrated forwards and backwards to an intermediate point, where a discrepancy exists. Does Eq. (7) converge? Zadunaisky & Pereyra (1965) have examined this problem and established sufficient conditions for convergence. Without going into mathematical detail, one can state heuristically that what may destroy convergence is: an initial approximation far from the final solution; an ill-conditioned matrix; the presence of large residuals. Because the $\mathbf{x}'s$ come from numerical integrations, large residuals arising from experimental error will not be present. Matrices from finite difference approximations are not highly ill-conditioned for reasonable stepsizes. Thus, only a poor initial approximation will destroy convergence. As long as the 1846 and the 1995–96 best fit orbits are not horribly bad, Eq. (7) should converge. Convergence, moreover, becomes more probable than the convergence of differential corrections, based on first-order Taylor series approximations to a nonlinear problem whereas Eq. (7) is based on tenth-order approximations.

One thus integrates the 1846 orbit forward to JD 2420000.5 and the 1995–96 orbit backwards to the same date and uses the rectangular coordinates from each orbit as the first approximation for Eq. (7). The coordinates for the perturbing bodies can be taken from a previous integration or from standard ephemerides. Because we need the comet's rectangular coordinates and are not calculating (O-C)'s, for which accurate coordinates for the Earth are required, we can use the Earth-Moon barycenter in Eq. (4). We can therefore take $h = 1^d$ in the equation. For comets that pass exremely close to the Earth, the barycenter lacks sufficent precision; one would have to use rectangular coordinates for the Earth and Moon separately, set $h = 0.^d25$, which changes the coefficients in Eq. (3) and leads to a matrix with four times as many unknowns, although still a band matrix. The starting coordinates for the two orbits furnish the boundary conditions for the problem. I used JD 2395314.5 and JD 2449995.5 as the boundary conditions. This results in a band matrix for 164,040 unknowns with bandwidth 5, equivalent to a 906×906 square matrix. Such a size is solvable on most personal computers. In fact, my computer, a 1 Ghz machine with 256 Mb of memory, calculates the solution in eight seconds. On the other hand, if I try to solve it as a square system, albeit triangular superior, the program aborts with the "out of memory" message.

But one must proceed with a certain amount of caution. Agreed, we can force the orbits to match at the fitting points, but how sensitive to the boundary conditions is the agreement at the fitting point? Use Eq. (7) to establish a sensitivity criterion. Let $\mathbf{x}_0$ be the left-hand boundary condition. The sensitivity of the solution $\mathbf{x}_{k+1}$ to the boundary condition, denoted by $\partial\mathbf{x}_{k+1}/\partial\mathbf{x}_0$, becomes

$$\mathbf{A} \cdot \partial\mathbf{x}_{k+1}/\partial\mathbf{x}_0 = \partial f(\mathbf{x}_k)/\partial\mathbf{x}_0 = (c_4\, c_3\, c_2\, c_1\, 0 \cdots 0\, c_4\, c_3 \cdots). \tag{8}$$

Upon rearranging Eq. (8) and taking norms of both sides we find

$$\| \partial \mathbf{x}_{k+1}/\partial \mathbf{x}_0 \| \leq \| \mathbf{A}^{-1} \| \cdot \| (c_4\, c_3\, c_2\, c_1\, 0 \cdots 0\, c_4\, c_3 \cdots) \| \, . \tag{9}$$

Thus, the sensitivity of the fitting point is determined by the norm of the matrix inverse $\mathbf{A}^{-1}$. As long as this norm is not unduly high, which it will not be for matrices approximating finite difference operators with reasonable stepsize, the fit should be satisfactory. One must keep in mind, nevertheless, that the boundary conditions cannot be arbitrary for the procedure outlined in this paper to work.

Figure 4 graphs the two solutions, the one from the 1846 observations and the one from the 1995–96 observations; the discrepancy at JD 2420000.5 is clearly visible. But the solution after the first iteration of Eq. (7) joins the orbits smoothly at the fitting point, as Figure 5 shows, although the orbit is not too close to the final orbit, also shown in Fig. 5. A total of five iterations produces a good orbit from which the initial conditions at JD 2420000.5,

$$\begin{array}{rr} x(AU) & -2.34614801134367 \\ y(AU) & -12.4149365728521 \\ z(AU) & -15.1097648537061 \\ \dot{x}(\mathrm{AU\ day}^{-1}) & 0.0005861839641 \\ \dot{y}(\mathrm{AU\ day}^{-1}) & 0.0029890882044 \\ \dot{z}(\mathrm{AU\ day}^{-1}) & 0.0020848962179 \end{array} .$$

The velocities come from a tenth-order formula for calculating the first derivative from function values derived in a similar way to the coefficients of Eq. (3).

TABLE 3

SOLUTION FOR RECTANGULAR COORDINATES AND VELOCITIES[a]

Unknown	Value	Mean Error
x_0	−2.3457004176998	0.0000451108129
y_0	−12.4144286322227	0.0000358451094
z_0	−15.1103388130149	0.0000379747086
$\dot{x}_0$	0.0005860946316	0.00000000902022
$\dot{y}_0$	0.00298900996560	0.0000000058739
$\dot{z}_0$	0.0020850106376	0.0000000075927
$\sigma(1)$	1.″89	

[a]For Epoch JD 2420000.5 and Equinox J2000.

TABLE 4

COVARIANCE (UPPER TRIANGLE) AND CORRELATION (LOWER TRIANGLE) MATRICES

24.2096	4.5321	−9.5019	−0.0047	−0.0007	0.0018
0.2356	15.2857	−15.6622	−0.0009	−0.0025	0.0032
−0.4662	−0.9672	17.1560	0.0018	0.0025	−0.0034
−0.9776	-0.2381	0.4537	0.0000	0.0000	0.0000
−0.2121	−0.9855	0.9591	0.1875	0.0000	0.0000
0.4300	0.9772	−0.9988	−0.4236	−0.9664	0.0000

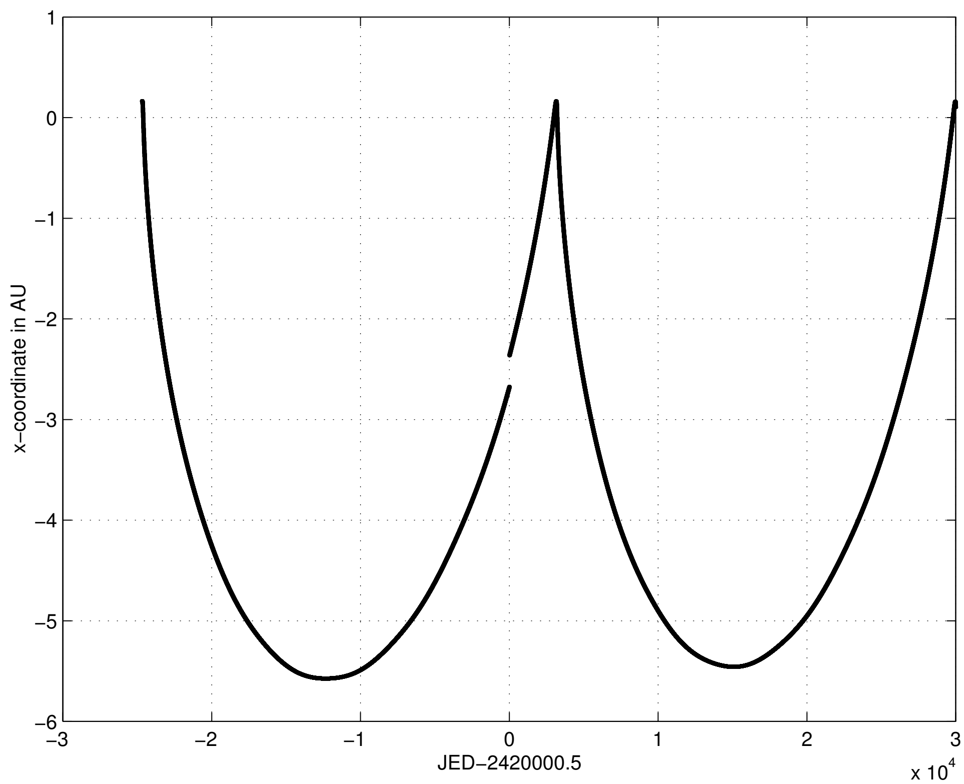

Fig. 4. Initial best-fit orbits.

TABLE 5

ELLIPTICAL ORBITAL ELEMENTS AND MEAN ERRORS[a]

Unknown	Value	Mean Error
M_0	319.°367359276965	0.°0000024549024
a	18.0186023009206	0.0000579424844
e	0.962972992857856	0.0000051049267
q	0.66717491608764	0.0000113464177
Ω	78.°3785836514897	0.°0003519217280
i	89.°2295500829416	0.°0003707365174
ω	230.°106440842196	0.°0012243068425

[a]For Epoch JD 2420000.5 and Equinox J2000.

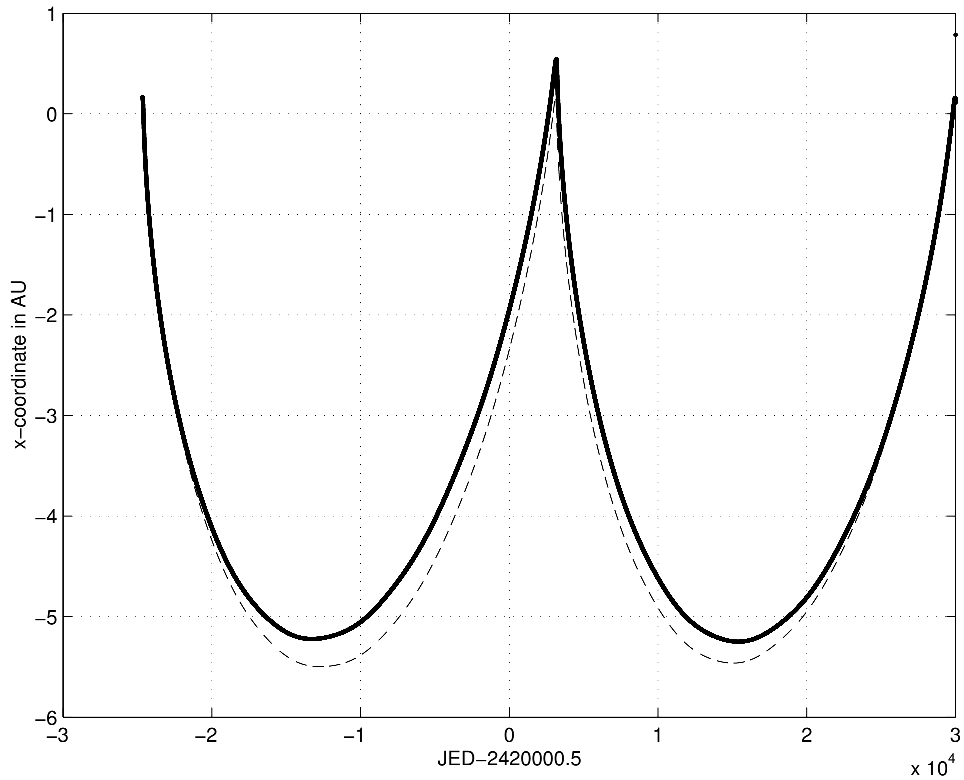

Fig. 5. Second (- -) and final (–) orbits.

6. THE SOLUTION

Table 3 shows the final solution for the rectangular coordinates, x_0, y_0, z_0, and velocities, $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$, along with their mean errors for epoch JD 2420000.5 and the mean error of unit weight, $\sigma(1)$. This Julian Date permits calculation of cometary position for the 1846 and the 1995–1996 observations without the greater accumulation of round-off of chopping error that would occur if one used a Julian Date close to either the 1846 or the 1995–1996 observations. Table 4 exhibits the covariance and correlation matrices for the solution. The correlations are high, and the singular value decomposition calculates a condition number of 6×10^7 for the matrix of the equations of condition, moderately high. This condition number is an unfortunate consequence of the poor distribution of the observations with respect to time; the condition numbers for the best fit orbits for the 1846 and the 1995–1996 observations taken separately are considerably lower, less than 10^3 in both instances, although the lower condition numbers imply little because these two best fit orbits do not permit linking the 1846 with the 1995–1996 observations.

Table 5 gives the orbital elements corresponding with the rectangular coordinates of Table 3: the mean anomaly of perihelion passage, M_0; the eccentricity, e; the major semi-axis, a; perihelion distance, q; the inclination, i; the node, Ω; and the argument of perihelion, ω. Rice's procedure (1902) was used to calculate the mean errors for the elliptical elements. To express Rice's procedure in modern notation let **C** be the covariance matrix for the least squares solution for the rectangular coordinates and velocities. Identify the errors in a quantity such as the node Ω with the differential of the quantity, $d\Omega$. The error can be found from

$$(d\Omega)^2 = \sigma^2(1)\left(\begin{array}{cccc} \partial\Omega/\partial x_0 & \partial\Omega/\partial y_0 & \cdots & \partial\Omega/\partial \dot{z}_0 \end{array}\right)\cdot \mathbf{C} \cdot \left(\begin{array}{c} \partial\Omega/\partial x_0 \\ \partial\Omega/\partial y_0 \\ \vdots \\ \partial\Omega/\partial \dot{z}_0 \end{array}\right). \tag{10}$$

The partial derivatives in Eq. (10) are calculated from the well known expressions linking elliptical orbital

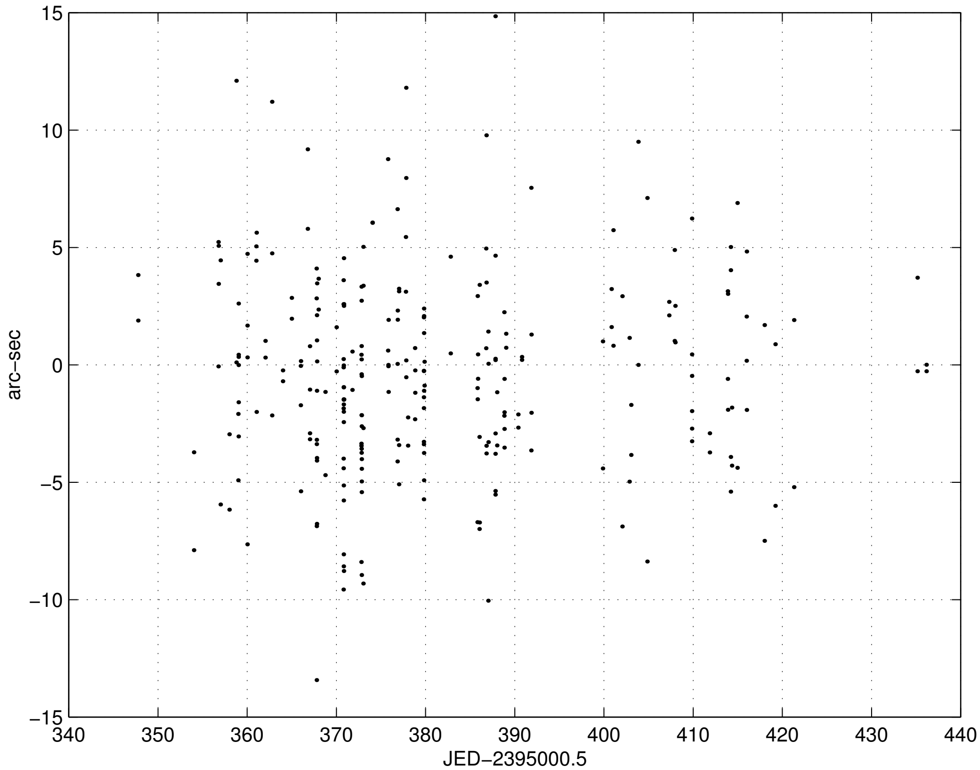

Fig. 6. Residuals for 1846 observations.

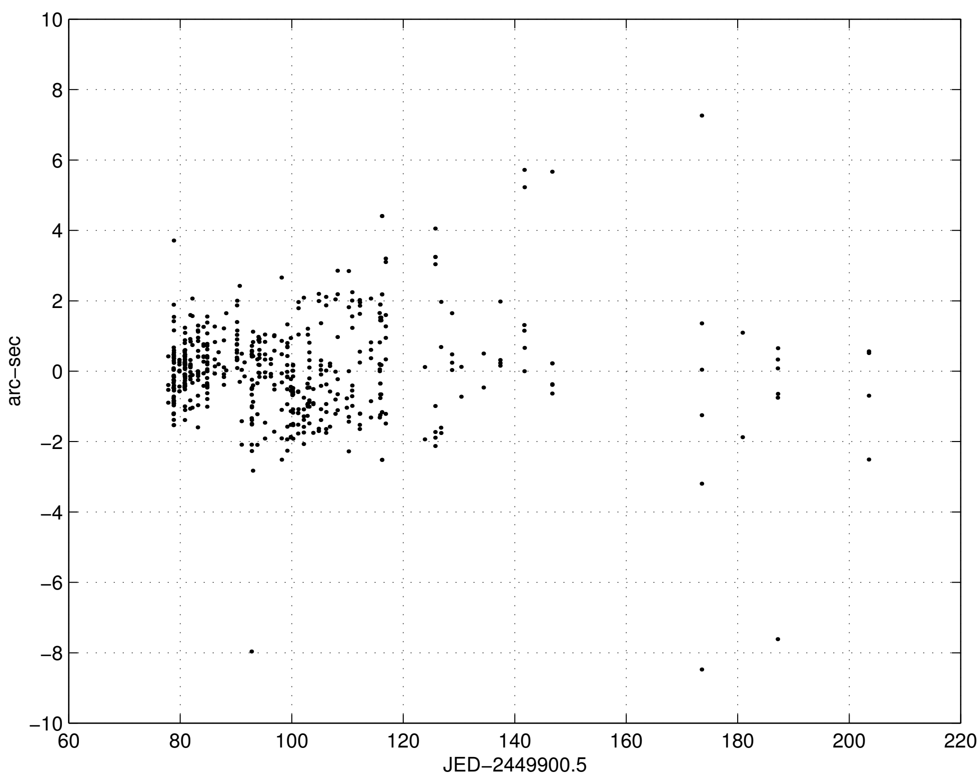

Fig. 7. Residuals for 1995–1996 observations.

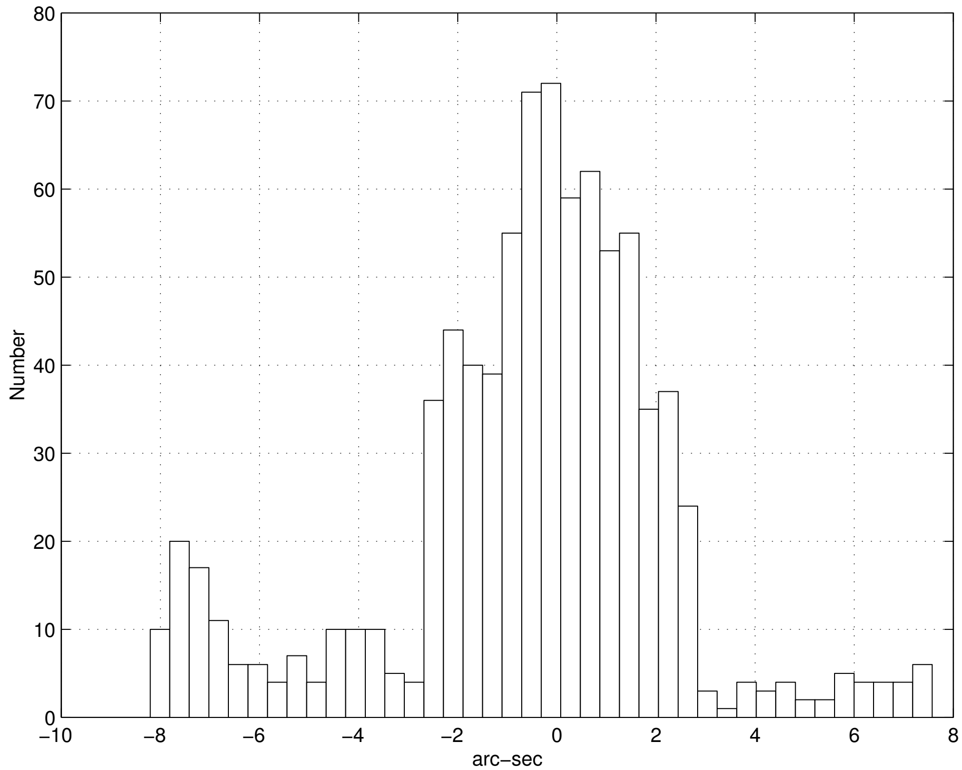

Fig. 8. Histogram of the residuals.

elements with their rectangular counterparts. Because the partials acquire great complexity, use of a symbolic manipulation language such as Maple becomes almost mandatory.

Figure 6 shows the residuals for the 1846 observations and Figure 7 the 1995–1996 residuals *before* the residuals were weighted by Eq. (1). The residuals are random: a runs test shows 381 runs out of an expected 398 with standard deviation of 20. Random, but not normal. Figure 8 shows a histogram of the residuals *after* weighting by Eq. (1). The residuals are skewed, factor of skewness = −0.329, and platykurtic, kurtosis= −0.211. But given that they are random—randomness is far more important for the statistical treatment of data than normality—one can consider the solution acceptable. The randomness also indicates that no unmodeled forces remain undetected, within the errors of the observations, of course.

From this orbit I calculated that de Vico should have had a perihelion passage during 1922 on 8.45792 Apr. Observations from that year would greatly strengthen the solution. But a literature search disclosed no observations that could be attributed to Comet de Vico during 1922. J. Calderón of the National Observatory of Córdoba, Argentina, offered to search through their old *Carte du Ciel* plates to see if any images possibly associated with the comet could be detected. I sent him an ephemeris, but the search proved sterile. Unfortunate!

The orbit also predicts that de Vico will appear again in 2069, with perihelion passage on 13.98761 Oct. Perhaps at that time more observations will be made to strengthen the orbit (although I doubt if *I* will be around to calculate the orbit) and to determine if the comet exhibits nongravitational effects.

7. CONCLUSIONS

An orbit for Comet 122P/ de Vico, based on all available observations, 400 in α and 396 in δ, is given. Linking the 19th century with the 20th century observations proved possible when the problem is treated as a boundary value problem. The linear system, although large with over 160,000 equations, is nevertheless banded and easily solved with square root-free Cholesky decomposition. Comet de Vico, not seen during the 1922 apparition, will next return in 2069.

I wish to thank Lic. Jesús Calderón and his collaborators at the Córdoba Observatory for their search of the *Carte du Ciel* plates.

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