Revista Mexicana de Astronomía y Astrofísica, **43**, 89–94 (2007)

TIDAL HEATING: A RESTRICTED THREE BODY PROBLEM INVESTIGATION OF THE SPITZER AND WEINBERG APPROXIMATIONS

Fidel Cruz and Luis A. Aguilar

Instituto de Astronomía
Universidad Nacional Autónoma de México, Ensenada, B. C., Mexico

Received 2005 September 13; accepted 2006 October 10

RESUMEN

Estudiamos el calentamiento por fuerzas de marea de cúmulos globulares utilizando el problema restringido de tres cuerpos. Nuestros resultados concuerdan más con la aproximación de Spitzer en el caso adiabático que con la de Gnedin y Ostriker para la aproximación de Weinberg. De estos resultados concluimos que la aproximación de Spitzer es válida para cúmulos en órbitas de alta excentricidad. Encontramos que un escalamiento lineal del calentamiento después de un choque de marea subestima el efecto que se mide después de varios choques.

ABSTRACT

We study the tidal heating of globular clusters using the restricted three body approximation. Our results are more consistent with the Spitzer correction for adiabaticity than with the Gnedin and Ostriker formula for the Weinberg correction. From these results we conclude that the Spitzer approximation is valid for clusters in orbits with high eccentricity. We find that a simple linear scaling of the effect after one tidal shock greatly underestimates the actual effect after repeated shocks.

Key Words: **CELESTIAL MECHANICS — GLOBULAR CLUSTERS: GENERAL — METHODS: NUMERICAL — STELLAR DYNAMICS**

1. INTRODUCTION

Contrary to what one may conclude from their isolated appeareance in the sky, globular clusters are shaped to a great extent by the galactic environment in which they move (Odenkirchen et al. 2003). The gravitational tidal field of the galaxy, in particular, enhances evaporation of stars and limits their spatial extent (Spitzer 1987).

In many studies of the effect of the environment on globular clusters, the three–body approximation has been an important tool and a substantial number of important results have been obtained with it. For example, Hénon (1970), found that retrograde orbits of test particles are more stable than direct orbits, in the Hill's limiting case of the restricted three body problem. Jefferys (1976) confirmed this result for the case of a Plummer model for the cluster potential and studied the degree of flattening in globular clusters produced by the galactic tidal field. Innanen (1979) studied the role of the Coriolis acceleration in stabilizing retrograde orbits and determined the relative sizes of tidal radii for stars in retrograde and direct orbits as a function fo the orbital eccentricity of the cluster.

Another important effect of the tidal field is the energy input produced by its time variation, when the cluster moves along an elongated orbit, or crosses the galactic disk. Ostriker, Spitzer, & Chevalier (1972) pioneered the study of these, so called, *tidal shocks*, for the case of galactic disk crossings. More recently, Chernoff, Kochanek, & Shapiro (1986) included this effect in a very thorough investigation of destruction processes that erode the population of globular clusters in our galaxy.

In most studies of globular cluster destruction to date, tidal heating has been modeled using an approximation introduced by Spitzer (1958). This approximation corrects the result obtained in the impulsive limit for adiabatic effects introduced when a finite, non-zero interaction time is considered. Spitzer's approximation models the motion of stars within a perturbed stellar system as forced one-dimensional harmonic oscilators, the forcing term being the external tidal field. In his original calculation, Spitzer computed the change in binding energy for stars inside a stellar cluster due to a passing perturbing mass. He found that the energy input into the stellar system decays exponentially as a function of the ratio of the perturbation time to the orbital period of the star, being maximum at the cluster's edge and quickly approaching zero toward its center. Aguilar, Hut and Ostriker (1988 hereafter AHO) used this approximation to compute the tidal heating of the globular clusters in our Galaxy due to the galactic bulge. They estimated that the current death rate is ~ 0.5 clusters per billion years.

More recently, Weinberg (1994) reviewed the original Spitzer approximation and introduced a refinement that proved crucial: he removed the one-degree of freedom constraint from the approximation. Adding extra degrees of freedom changes qualitatively this problem: the perturbation can resonate with linear combinations of the basic frequencies of the stars along the various degrees of freedom, resulting in enhanced heating. Contrary to the Spitzer exponential dependence, Weinberg found a power law behavior that decreases with decreasing radial position within the cluster, resulting in a more important tidal heating. Gnedin and Ostriker (1997, hereafter GO) incorporated the Weinberg revision in their reanalysis of the AHO investigation of cluster destruction rates. They found a destruction rate 2 to 5 times higher than the result of AHO. This discrepancy arises from most detailed modeling of cluster dynamics with the Fokker-Planck calculation.

We propose to study with a simple model the adiabatic corrections of Spitzer and Weinberg, using a set of three-body experiments to verify the validity of this approximations for tidal heating within the context of globular cluster destruction. This approximation does not include self-gravity of cluster stars and its virialization after cluster shock, that has a very important effect on the energy distribution (Gnedin & Ostriker 1999). However, a direct comparison of this approximations with this simple model study will give us a better comprehension of tidal shocks. In §2 we briefly review the Spitzer and Weinberg approximations. In §3 we describe the numerical experiments that we made. Results of our numerical simulation and their comparison with the Spitzer and Weinberg approximations are presented in §4 and the implications of our results for studies of tidal heating of globular clusters are presented in §5.

2. THE ANALYTICAL APPROXIMATIONS

Spitzer considered a point mass (the star inside the cluster) moving within a one-dimensional harmonic potential (the cluster) and subject to an external force produced by a point mass perturber moving at uniform speed along a straight path (the galaxy). He solved the equations of motion using the variation of coefficients method and found that the change in energy is given by:

$$\Delta E = \frac{1}{3}\left(\frac{2GM_p}{vp^2}\right) 2r^2 \eta_S(\beta) \ , \qquad (1)$$

where G is the gravitational constant, M_p is the perturber's mass, v is the relative velocity between the perturber and the star, p is the minimum distance between them and r is the orbital radius of the star. Spitzer's correction for adiabaticity η_S is given by,

$$\eta_S(\beta) = \frac{1}{2}\left[L_x(\beta) + L_y(\beta) + L_z(\beta)\right] \ , \qquad (2)$$

here the L_i are linear combinations of modified Bessel functions (Spitzer 1958). The argument β, is called the adiabatic parameter and it is proportional to the ratio of the perturbation time to the period of the star:

$$\beta \equiv \frac{2\omega_s p}{v} \ , \qquad (3)$$

where ω_s is the orbital frequency of the star.

Weinberg has argued that the adiabatic model used in the Spitzer approximation is highly artificial. A one-dimensional oscillator is unrealistic even for a star in orbit in a spherical potential: the orbits are planar with radial and azimuthal frequencies. The presence of various degrees of freedom changes the problem in a non-trivial way. Each degree of freedom introduces its own frequency and, for the perturbation to be adiabatic, its associated frequency must be small not only compared to each independent frequency, but to all integer linear combinations of them. Even if the perturbation occurs slowly compared to each degree of freedom taken independently, there may be still be a resonance that pumps energy into the system at a rate not predicted by the one-dimensional model.

Weinberg's correction for eq(1) has been approximated analytically by GO as:

$$\eta_W(\beta) = \left(1 + \frac{\beta^2}{4}\right)^{-3/2} \ , \qquad (4)$$

where β is given by equation 3 above. Both adiabatic corrections (equations 2 and 4) approach 1 (the impulsive limit) for $\beta \longrightarrow 0$, and decay to zero for large values of it (adiabatic limit), although η_S shrinks exponentially while η_W does so as a power law.

AHO considered two additional corrections: One for a perturber with an extended mass distribution and another for the cluster orbital eccentricity. For comparison with our numerical experiments, it is only the second correction that is relevant, as we consider only point masses. In this case, AHO's second correction (their equation 11) becomes

$$\lambda(e) = \left[\left(\frac{1-e}{1+e}\right)^3 - 1\right]^2 , \qquad (5)$$

where λ is a multiplicative factor that must be incorporated in equation 1. Its value increases with orbital eccentricity and in this case, p is set equal to the perigalacticon distance.

3. THE NUMERICAL EXPERIMENTS

We have made a series of three-body experiments to verify the validity of the Spitzer and Weinberg corrections for adiabaticity, for stars in globular clusters subject to the tidal heating of the galactic bulge. We model the galactic bulge and the cluster as point masses in orbit around each other. We choose a system of units such the the gravitational constant and the cluster mass are equal to unity, while the cluster perigalacticon is equal to 400. In these units, the galactic bulge has a mass equal to 10^4. We use this value as it is similar to the one used by AHO and GO.

Each experiment is defined by the cluster orbit eccentricity (e =0.00, 0.25, 0.50, 0.75) with the same perigalacticon, as we define above. For each experiment we consider an ensemble of test particles moving initially on circular orbits around the cluster, at various spatial orientations and with different initial phases. A test particle orbital plane is defined by two angles: We have used a 4×6 uniform mesh for them. For each orbital plane, we consider 40 initial phases, and for each, we consider two senses of rotation around the cluster center to define direct and retrograde orbits. Additionally, we have considered 100 initial orbital sizes on each orbital plane, logarithmically spaced in the range $r = 3.16$ to 10.0 in our units; these correspond to a range of adiabatic parameter of $\beta = 28.4$ to 5.0, for the circular cluster orbit case. This results in an ensemble of 192,000 test particles for each experiment.

As a guide in interpreting these experiments, we note that if we adopt a mass of $5 \times 10^5\, M_\odot$ for the cluster and 2 kpc for its perigalacticon the corresponding apogalactica are 3.3, 6 and 14 kpc. The galactic bulge has a mass of $5 \times 10^9\, M_\odot$ and the range of radial distances for the stars within the cluster that we consider is 16 to 50 pc. The theoretical tidal radius (Innanen, Harris, & Webbink 1983) for a cluster in circular orbit is 49 pc in this case. In our units the tidal radius corresponds to ≈ 9.6.

The orbits are integrated using a Burlisch-Stoer integrator (Press et al. 1992) in a double precision FORTRAN code. In order to determine the accuracy of this code we verified the Jacobi energy conservation for 10 cluster periods for the $e = 0$ case, and found to be good to to one part in 10^{16} per timestep or better, which corresponds to 1/100 of the test particle period.

We compute the binding energy of the stars with respect to the cluster as follows;

$$E_s = \frac{1}{2}v_{cs}^2 - \frac{M_c}{r_{cs}} , \qquad (6)$$

where v_{cs} is the velocity of the star with respect to the cluster, M_c the cluster mass, and r_{cs} the distance between cluster and corresponding star.

We compute the average heating for all the stars with the same initial radius as follows,

$$< \Delta E_{cs} >= \frac{1}{N_s}\sum_{i=1}^{N_s}(E^i - E_0^i) , \qquad (7)$$

where E_0^i and E^i are the initial and final energy (as defined in equation 6) for the i test particle and the summation is over the N particles at the same initial radius.

We note that the energy of each particle, as defined by equation (6), is not conserved, even for the case of clusters in circular orbits. We need take note that this is not the Jacobi constant or the total binding energy within the combined galaxy and cluster potential. This energy correspond to the cluster star energy whose change we are interested in. In our result, we will use the circular orbit case as our reference level. Figure 1 shows the energy differences computed for the three eccentricities studied. In each case, only test particles in retrograde orbits are considered. We also show the corresponding circular orbit case.

4. RESULTS

Figure 2 shows the change in binding energy for test particles in retrograde orbits within the cluster, as a function of the parameter β (as a guide, the upper axis shows the equivalent r/r_t, computed at

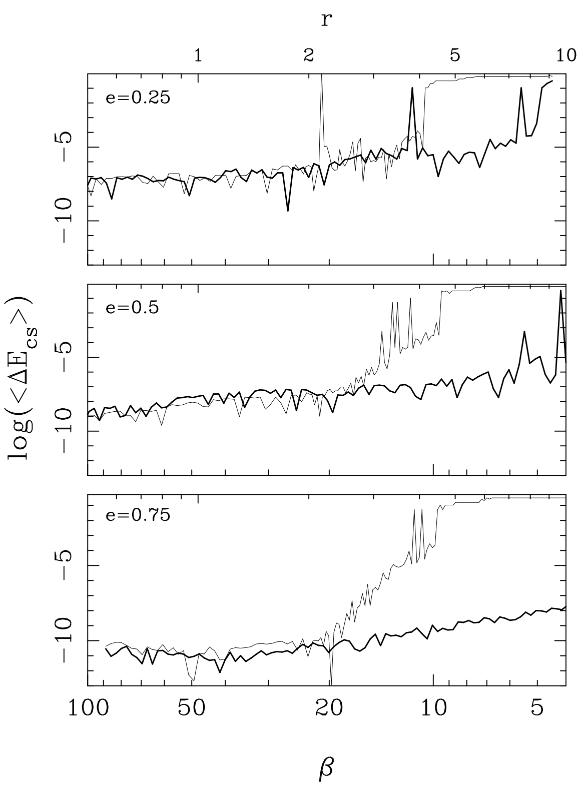

Fig. 1. Energy change after ten cluster orbital periods as a function of position within the cluster for the retrograde test particles (thin lines). In each frame, the results for the circular orbit at the corresponding apogalactica are shown too (thick lines). We have used β for easy comparison with the following figures. Remember that β has no meaning for clusters in circular orbits. The corresponding radial positions are shown in the upper axis of the figure.

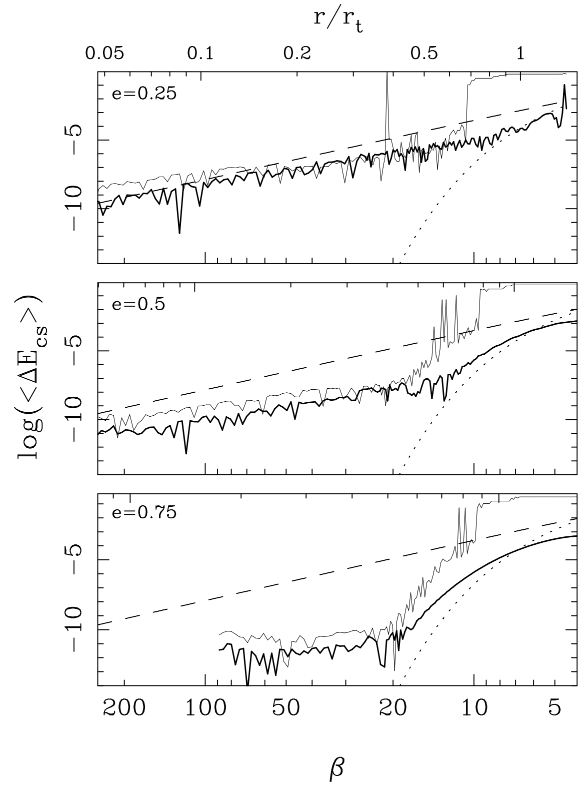

Fig. 2. Energy change after one (thick line) and ten cluster orbital periods (thin line) for the retrograde test particles. Each panel show results for the cluster orbital eccentricity indicated in the upper left corner. The dotted and dashed lines are the Spitzer and Weinberg approximation, respectively.

perigalacticon). The thick lines show the change after just one cluster orbit, whereas the thin lines show the accumulated change after ten cluster orbits. The three panels correspond to cluster orbital eccentricities of 0.25 (upper panel), 0.50 (middle panel) and 0.75 (lower panel). The dotted and dashed lines in all panels are the Spitzer and Weinberg approximation from Eq. 1 with Eq. 2 and Eq. 3, respectively, using the normalization relevant to these simulations.

The results after just one cluster period show two trends, a power-law regime at small r and an exponential behavior at large r. The first coincides to a large extent with the results of the circular orbit case and it is to be dismissed (see §3), the second corresponds to the tidal heating we want to study. As the orbital eccentricity of the cluster increases, the effect of tidal heating gets deeper within the cluster reaching $\beta \sim 20$ for our most eccentric case. We also note that the trend is exponential, like in Spitzer's correction, and not power-law, like in Weinberg's. The power-law regime has a different slope for each eccentricity. This also was observed by Gnedin & Ostriker 1999. They found that the energy variation could be fitted by the same functional but with a different exponent. However, the Spitzer approximation fitted their results in the external part of the cluster, like in our case.

Moving now to the accumulated effect after ten cluster orbital periods we see that a sharp step is introduced close to the tidal radius ($\beta \sim 7$). This is due to a population of very loosely bound particles that is being eroded continuously after each perigalacticon passage. This behavior is not described by any of the adiabatic corrections; however, its contribution to the overall heating of a cluster will be small since for reasonable concentration (c=1.5), less than 4% of the total mass of a cluster is within this region ($r \gtrsim 0.7 r_t$).

One important result, readily apparent in Figure 2, is the failure of estimating the amount of tidal heating after repeated perigalacticon passages as a simple linear scaling of the Spitzer approximation after just one passage. If we center our attention in the intermediate region where most of the heating takes place ($20 > \beta > 10$), we notice that shifting the thick lines in each panel upwards by a distance equal to one tick mark in the vertical axis (a factor of 10) does not get us to the thin lines, which are the measured effect after ten cluster periods. In fact, the effect is larger.

In Figure 3 we show results for prograde and retrograde test particles orbits for the $e = 0.5$ case during the first four cluster orbital periods. Here we can clearly see the secular heating produced by repeated perigalacticon passages and the appearance of the loosely bound population near the tidal radius. However, there is a systematic difference between prograde and retrograde orbits: the former get more efficiently heated and develop the nearly unbound population of particles after just one passage. This is consistent with previous studies. For instance, Innanen (1979) found this effect and argued that it is due to the Coriolis term in the equation of motion, that opposes the cluster gravitation for direct orbits while it reinforces it for retrograde orbits. After only four periods this difference has disappeared, as the amount of heating becomes large near the tidal radius.

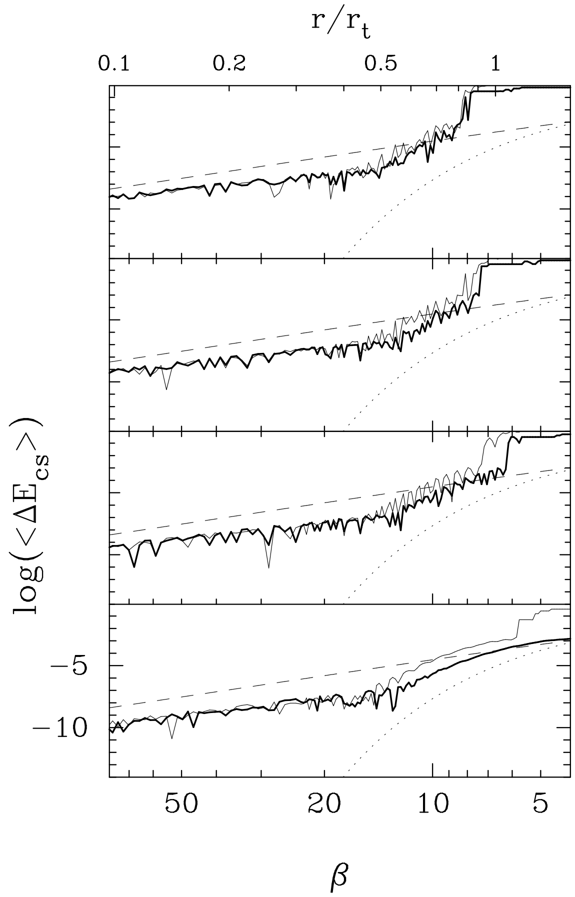

Fig. 3. Change in the binding energy from the first (bottom panel) to the fourth (top panel) orbital periods for direct (thin line) and retrograde (thick line) orbits, for our experiments with e=0.50. Dashed and dotted lines are the Spitzer and Weinberg approximation, respectively.

5. DISCUSSION AND CONCLUSIONS

We have studied in detail the mechanism of tidal heating on a cluster in bound orbit around the Galaxy using the three-body problem. Although highly idealized, this approach offers us a simple way to study some basic aspects of this mechanism. For a star moving in the combined potential of a Galaxy and cluster, care must be taken to define the energy used to gauge the amount of tidal heating, as it is only the Jacobi energy that is conserved, and this is only in the circular orbit case. In our case, we have used the circular orbit case to define our zero-heating standard of reference.

We have found that the amount of tidal heating varies exponentially (like Spitzer's adiabatic correction) and not as a power-law (like in GO interpretation of Weinberg correction), at least we cannot differentiate it from the energy variation with respect to the circular orbit. Most of our results show the same behavior as the N-body simulations by Gnedin & Ostriker (1999), where radial experiments show an exponential variation in the external part of the cluster and almost a flat energy variation inside the cluster.

At fixed perigalacticon, the amount and the depth of the tidal heating effect increases with increasing cluster orbital eccentricity, as we move to the radial case. Particles on direct orbits are more readily affected than those on retrograde ones.

Although the Spitzer approximation gives an adequate estimate for the amount of tidal heating after one tidal shock, its linear scaling is not appropriate when applied to repeated shocks. For weak shocks, after each shock the stars in the cluster change their original periods and move steadily towards the impulsive region. Strong shocks strip stars from the cluster and make subsequent shocking inefficient as we show in §4 where the mass outside 0.7 r_t is less than 4% and is diminished at each perigalactic passage. This process has ben studied in detail by Gnedin, Lee, & Ostriker (1999). This must be taken into account, otherwise the estimated effect will fall short of the actual heating.

Our work does not address effects due to extended mass distribution for the Galaxy and the cluster, or effects due to tidal distortion in the cluster. We will address these questions in a follow-up study using full N-body simulations.

We thank the anonymous referee for several suggestions that improved this paper.

REFERENCES

Aguilar, L., Hut, P., & Ostriker, J. P. 1988, ApJ, 335, 720 (AHO)
Chernoff, D. F., Kochanek, C. S., & Shapiro, S. L. 1986, ApJ, 309, 183
Gnedin, O. Y., Lee, H. M., & Ostriker, J. P. 1999, ApJ, 522, 935
Gnedin, O. Y., & Ostriker, J. P. 1997, ApJ, 474, 223 (GO)
_________. 1999, ApJ, 513, 626
Hénon, M. 1970, A&A, 9, 24
Innanen, K. A. 1979, AJ, 84, 960
Innanen, K. A., Harris, W. E., & Webbink, R. F. 1983, AJ, 88, 338
Jefferys, W. H. 1976, AJ, 81, 983
Odenkirchen, M., et al. 2003, AJ, 126, 2385
Ostriker, J. P., Spitzer, L., & Chevalier, R. A. 1972, ApJ, 176, L51
Press, W. H., Flannery, B. P., Teukolsky, S. A, & Vettering, W. T. 1992, Numerical Recipes in Fortran 77: Art of Scientific Computing (Cambridge: Cambridge Univ. Press)
Spitzer, L. 1958, ApJ, 127, 17
_________. 1987, Dynamical Evolution of Globular Clusters (Princeton: Princeton Univ. Press)
Weinberg, M. D. 1994, AJ, 108, 1398

Luis Aguilar and Fidel Cruz: Instituto de Astronomía, Universidad Nacional Autónoma de México, Apdo. Postal 877, 22860 Ensenada, B. C., Mexico (aguilar, fidel@astrosen.unam.mx).