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# AMPLITUDE FLUCTUATIONS IN CURVATURE SENSING: COMPARISON OF TWO SCHEMES

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RESUMEN

Se investiga la influencia de las fluctuaciones en amplitud sobre la calidad de la reconstrución de fases en la medición de la curvatura. Se comparan los dos esquemas: el que emplea dos imágenes simétricas fuera de foco (esquema de Roddier) y el que emplea una sola (esquema de Hickson). Se demuestra que la precisión de la reconstrucción de fases con el esquema de Hickson se ve fuertemente afectada por fluctuaciones en amplitud incluso leves, mientras que el esquema de Roddier funciona bien incluso con grandes fluctuaciones en amplitud.

ABSTRACT

The influence of amplitude fluctuations on the quality of phase reconstruction in curvature sensing is investigated. Two curvature schemes are compared: one with two symmetrically-defocused images (Roddier scheme) and one with a single defocused image (Hickson scheme). It is shown that the accuracy of phase reconstruction with the Hickson scheme is strongly affected even by low-level amplitude fluctuations, while the Roddier scheme works well under quite strong amplitude fluctuations.

*Key Words:* instrumentation: adaptive optics — techniques: high angular resolution

## 1. INTRODUCTION

Curvature sensing is a method for phase measurements suggested by F. Roddier (Roddier 1987). The method allows for phase reconstruction using two symmetrically-defocused images, and its experimental setup is shown schematically in Figure 1.

The distorted input wavefield passes the lens $L_1$ with focal length $f$. The curvature signal $\eta(\vec{r})$ is composed of the intensity distributions $I_1$ and $I_2$ of two symmetrically-defocused images. The focal length of the lens $L_2$ is equal to $f/2$. The phase $S(\vec{\rho})$ of the input wavefield can be retrieved from the curvature signal solving the following equation

$$\eta(\vec{r}) = -\frac{f(f-l)}{kl}\left[\triangle S(\vec{\rho}) - \delta(\vec{\rho}-R)\frac{\partial S(\vec{\rho})}{\partial \vec{\rho}}\right],$$
$$\eta(\vec{r}) = \frac{I_2(\vec{r}) - I_1(-\vec{r})}{I_2(\vec{r}) + I_1(-\vec{r})}, \tag{1}$$

where $R$ is the aperture radius, $k$ is the wavenumber, $\delta$ denotes the delta-function, and the right-side functions are evaluated for $\vec{\rho} = f\vec{r}/l$.

Equation 1 has been derived applying the geometric optics approximation to the Fresnel integral (Roddier 1987) and it is valid when:

$$\frac{\lambda(f-l)}{r_0} \ll \frac{r_0 l}{f}, \tag{1a}$$

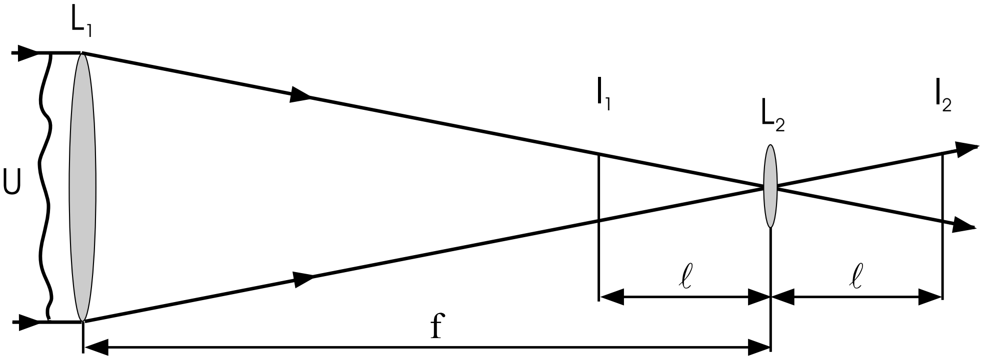


Fig. 1. Experimental setup of curvature sensing.

where $r_0$ is the correlation length of the wavefront fluctuations (Fried's parameter), $f$ is the focal length, and $\lambda$ is the wavelength (Roddier 1988).

A modification of the Roddier scheme has been proposed by Hickson (Hickson 1994) who suggested the use of a single defocused image (either intrafocal or extrafocal) instead of two symmetrically-defocused ones. In this case, the input phase is retrieved from the same equation 1, but the curvature signal $\eta(\vec{r})$ is obtained as $[I_1(\vec{r}) - I_0]/I_0$, where $I_0$ is a long-exposure intensity at the image plane. From the technical point of view the Hickson scheme is more attractive because, compared to the Roddier scheme, it assumes a significantly simplified setup.

Equation 1, which allows the reconstruction of the phase from the curvature signal, is derived for the case of uniform illumination. However, in real experiments we never have such perfect conditions, so it is important to know how strongly amplitude fluctuations affect the quality of phase retrieval. In this paper we use computer simulations to compare the ability of phase retrieval for both schemes in the case of non-uniform pupil illumination.

## 2. SIMULATION METHOD

To treat the problem of interest by means of simulations we have to calculate numerically a curvature signal for a given wavefield at the lens aperture. According to the Fresnel approximation, the field $U(\vec{r})$ at the out-of-focus image plane of a lens can be described as (Born & Wolf 1987)

$$U(\vec{r}) = \frac{k\,\exp(ikz)}{2\pi iz}\int_{G_R} d^2\rho A(\vec{\rho})\,\exp[iS(\vec{\rho})]\,\exp\left[i\frac{k}{2z}\left|\vec{\rho}-\vec{r}\right|^2\right]\,\exp\left[-i\frac{k}{2f}\rho^2\right], \tag{2}$$

where $U(\vec{r})$ is the field at the image plane, $A(\vec{\rho})$ and $S(\vec{\rho})$ are the amplitude and the phase at the lens aperture, $z$ is the distance from the aperture to the image plane, $k$ is the wavenumber, $f$ is the focal length of the lens. $G_R$ means that the integration is performed over the aperture plane, and $\vec{\rho}$ and $\vec{r}$ denote the two-dimensional position vectors at the aperture and image plane, respectively.

In what follows we use the Zernike polynomial expansion for the initial phase

$$S(\vec{\rho}) = k\sum_{l=2}^{L} a_l Z_l\left(\frac{\vec{\rho}}{R}\right), \tag{3}$$

where $Z_l$ denotes the $l$-th two-dimensional Zernike polynomial (Noll 1976), and $R$ stands for the aperture radius.

Substituting (3) into (2), we get

$$U(\vec{r}) = \frac{k\,\exp(ikz)}{2\pi iz}\int_{G_R} d^2\rho A(\vec{\rho})\exp[ikf(\vec{\rho})],$$
$$f(\vec{\rho}) = \left[\frac{\left|\vec{\rho}-\vec{r}\right|^2}{2z} - \frac{\rho^2}{2f} + \sum_{l=2}^{L} a_l Z_l\left(\frac{\vec{\rho}}{R}\right)\right]. \tag{4}$$

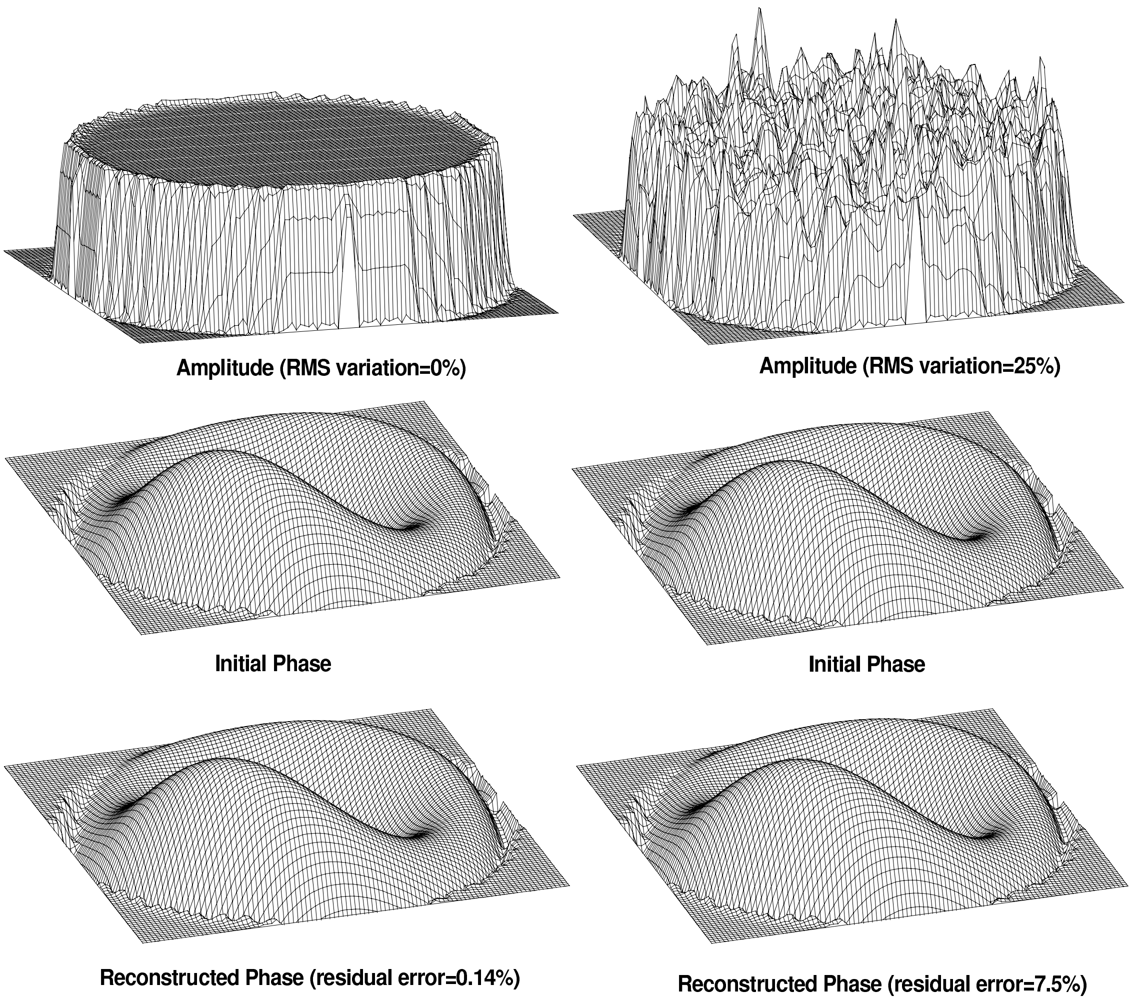


Fig. 2. Effect of amplitude fluctuations for the Roddier scheme (two symmetrically-defocused images).

The integral (4) can be evaluated using its asymptotic expansion (Wong 1989) as

$$U\left(\vec{r}\right)=\frac{\exp\left(ikz\right)}{z}\sum_{n=1}^{N}\frac{\sigma_n}{\sqrt{|\alpha_n\beta_n-\gamma_n^2|}}A\left(\vec{\rho}_n\right)\exp\left[ikf(\vec{\rho}_n)\right]\,, \tag{5}$$

where $N$ is the number of critical points, and $\vec{\rho}_n=\{x_n,y_n\}$ are the Cartesian coordinates of the vector $\vec{\rho}_n$.

The coordinates $\vec{\rho}_n$ of the $n$-th critical point are determined from the equation

$$\frac{\partial f(\vec{\rho}_n)}{\partial x}=\frac{\partial f(\vec{\rho}_n)}{\partial y}=0\,, \tag{6}$$

while the quantities $\sigma_n,\alpha_n,\beta_n$, and $\gamma_n$ are given by

$$\alpha_n=\frac{\partial^2 f(\vec{\rho}_n)}{\partial x^2}\,,\quad \beta_n=\frac{\partial^2 f(\vec{\rho}_n)}{\partial y^2}\,,\quad \gamma_n=\frac{\partial^2 f(\vec{\rho}_n)}{\partial x\partial y}\,, \tag{7}$$

$$\sigma_n=\begin{cases}\ \ 1,\alpha_n\beta_n-\gamma_n^2>0\,,\alpha_n>0,\\ -1,\alpha_n\beta_n-\gamma_n^2>0\,,\alpha_n<0,\\ -i,\alpha_n\beta_n-\gamma_n^2<0\,.\end{cases} \tag{8}$$

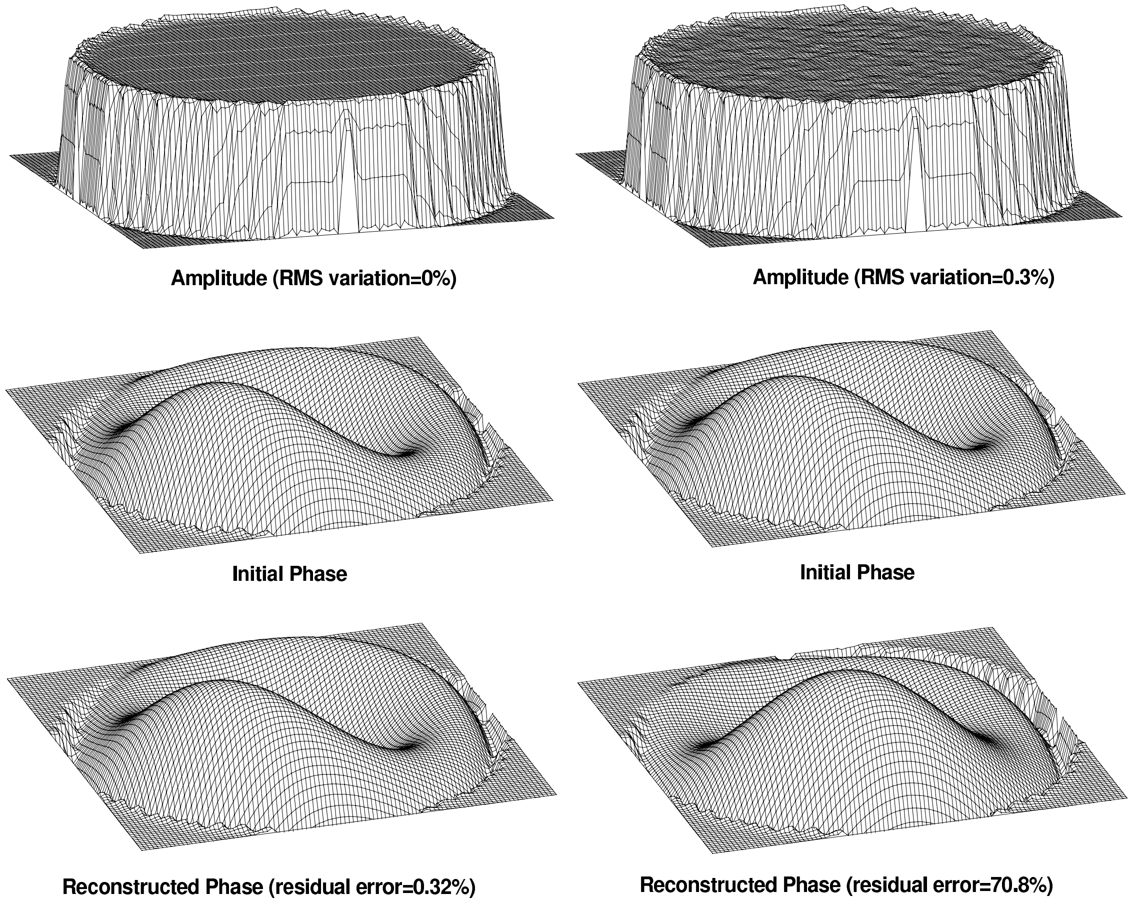


Fig. 3. Effect of amplitude fluctuations for the Hickson scheme (a single defocused image).

The approach of asymptotic expansion is often used for numerical evaluations of quickly-oscillating integrals, as it allows to avoid time-consuming calculations on big grids. In particular, this method is suitable for the evaluation of Fresnel integral (4) (Born & Wolf 1987). Equation 5 presents only the first terms of the expansion that correspond to the case of geometric optics. For our considerations this restriction is important because it has the same range of validity as the curvature sensing equation 1.

Taking into account the above considerations, the simulation procedure is as follows. First, we simulate some amplitude $A$ and phase $S$ distortions at the aperture. Then, using equations 5–8, we evaluate numerically the out-of-focus intensity distributions for a given input field. As soon as the needed intensities are obtained we compose the curvature signal corresponding to the chosen scheme. Finally we retrieve the phase $S_r$ solving equation 1, and we compare the retrieved phase $S_r$ to the initial one $S$.

To solve equation 1 we use the following method. As one can see from equation 1, the curvature signal consists of two terms: the first one is the phase Laplacian $\triangle S$ (the curvature part of the signal), while the second one $\delta\left(\vec{\rho}-R\right)\frac{\partial S(\vec{\rho})}{\partial\vec{\rho}}$ describes the singular boundary signal. In reality the boundary signal magnitude is finite and it is located inside a thin but finite edge zone, while the singularity appears because equation 1 has been derived in the geometric optics approximation. However, such a singularity does not allow us to define properly the influence of amplitude fluctuations on the boundary signal, so we restrict our attention to the curvature part of the signal. Under this condition, and taking into account equation 3, we can rewrite equation 1 as

$$\eta\left(\vec{r}\right) = -\frac{f\left(f-l\right)}{l}\sum_{l=2}^{L} a_l \Delta Z_l\left(\frac{\vec{\rho}}{R}\right) . \tag{9}$$

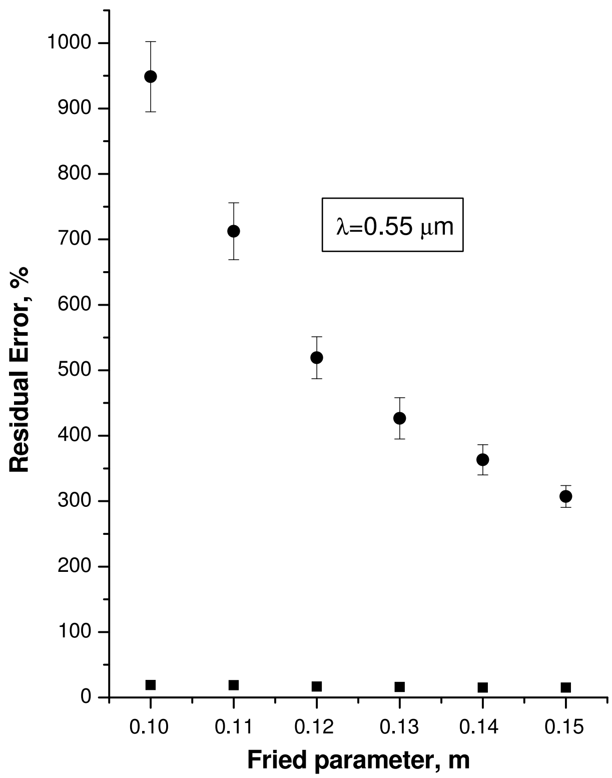


Fig. 4. Residual error of phase reconstruction for V-band. The scintillation-less case is plotted by squares.

Representing the phase $S$ as a linear superposition of Zernike polynomials we can solve equation 9 by the method of least-squares:

$$\int_{G_R} d^2\rho \left[\eta\left(\vec{r}\right) + \frac{f\left(f-l\right)}{l}\sum_{l=2}^{L} a_l \Delta Z_l\left(\frac{\vec{\rho}}{R}\right)\right]^2 = \min\ . \tag{10}$$

Equation 10 reduces the problem to the numerical solution of the system of linear algebraic equations that give the values of coefficients $a_l$. As a result, we get the retrieved phase as a linear superposition of Zernike. polynomials

## 3. SIMULATION RESULTS

Using the approach described in the previous section, we have made some simulations which show how amplitude fluctuations affect the quality of phase retrieval for the two schemes. The simulations are performed for the following parameters: the aperture diameter is 1m, the focal length is 15m, the out-of-focus distance is 0.1m, and the wavelength is 0.55 $\mu$m.

The phase distortions are simulated as a linear superposition of Zernike polynomials with randomly chosen coefficients, while the amplitude fluctuations are simulated as a random Gaussian noise. The main aim of these simulations is just to compare the effect of interest in the two schemes, rather than a detailed quantitative analysis. For this reason we present some typical results in two figures below.

Figure 2 illustrates the effect of amplitude fluctuations in the Roddier scheme.

The first column of images presents the case of uniform illumination (the initial amplitude is constant, as shown in the first image). The second image shows the initial phase while the third one demonstrates the restored phase. As one can see, the residual error of reconstruction defined as the normalized r.m.s. difference between the initial and restored phase is quite small (0.14%). This error is due to the numerical calculations and its very small value demonstrates the validity of the simulation procedure.

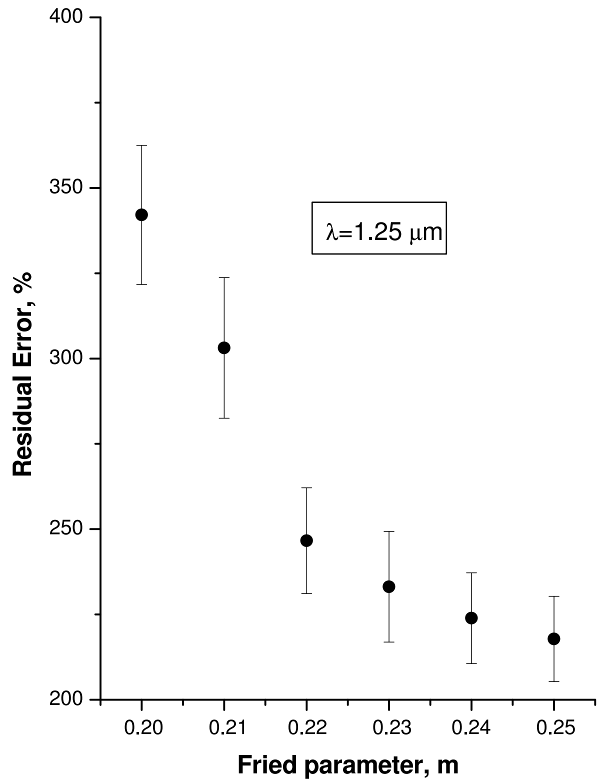


Fig. 5. Residual error of phase reconstruction for J-band.

The second column in Figure 2 shows a case of non-uniform illumination. One can see that despite a very high level of amplitude fluctuations (25%) the phase reconstruction error is not large (7.5%). This result is not a surprise, because the differential character of the scheme effectively suppresses the effect of amplitude fluctuations on the curvature signal.

Figure 3 illustrates the effect of amplitude fluctuations in the Hickson scheme.

As in the case of Roddier scheme, the first column presents the results for uniform illumination. One can see that the phase is restored well (the error is equal to 0.32%). However, even in the case of quite small amplitude fluctuations, the phase reconstruction becomes practically impossible. This is illustrated in the second column of Figure 3. One can see that a 0.3% level of amplitude fluctuations results in a 70.8% level of reconstruction error.

Now, let us consider the effect of turbulence-induced amplitude fluctuations (scintillations). Because the influence of scintillations on the Roddier scheme has already been studied in (Voitsekhovich & Sánchez 2003), we restrict here our attention to the case of the Hickson scheme.

Compared to the situation considered above, the case of distortions induced by atmospheric turbulence is quite different, because amplitude and phase fluctuations are cross-correlated and obey certain statistics. A method that allows one to simulate cross-correlated amplitude and phase fluctuations produced by atmospheric turbulence is described in Kouznetsov, Voitsekhovich, & Ortega-Martínez (1997). Using this method we simulate the phase and amplitude samples at the aperture and restore the phase for the case of the Hickson scheme. The simulation parameters are as follows: the aperture diameter is 1m, the focal length is 15m. The simulations are performed in two wavelengths ($V$-band, 0.55 $\mu$m, and $J$-band, 1.25 $\mu$) for different Fried parameters $r_0$ (varying from 0.1m to 0.25m). The out-of-focus distance $l$ is chosen from the equation $l = 10\frac{\lambda f^2}{\lambda f + r_0^2}$ that comes from equation 1a. This choice of $l$ guarantees that the simulations are performed inside the working range of the conventional (geometric-optics based) curvature sensing. Other choices of $l$ are not investigated. Also, because in this paper we study the method of the curvature sensing itself rather than its applications for adaptive optics, the study of the Hickson scheme under closed-loop conditions is not performed.

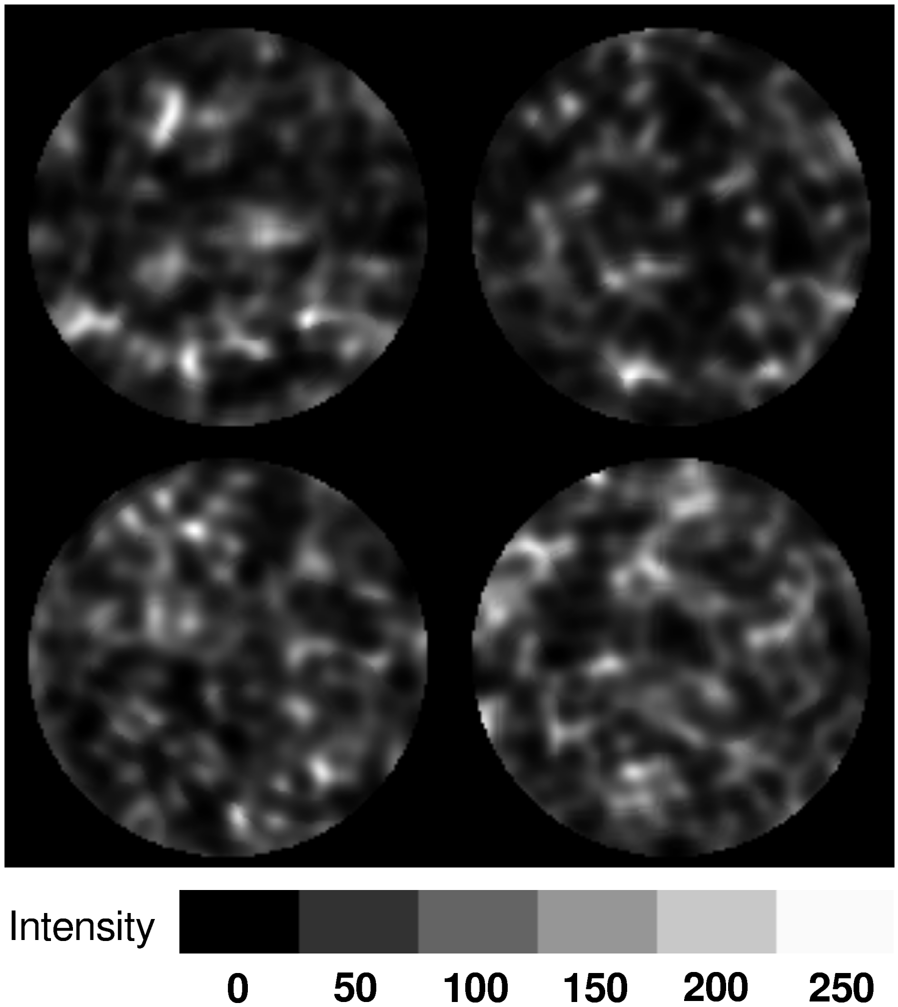


Fig. 6. Samples of turbulence-degraded intrafocal images. The Fried parameter $r_0 = 0.15$ m, the wavelength $lambda = 0.55$ $\mu$m, the out-of-focus distance $l = 0.12$ m. Note: because the simulations are performed in the geometric optics approximation, diffraction-related effects (like fuzzy edges of the pupil, etc.) do not appear.

The simulation results are shown in Figures 4 and 5. Every point in the graphs is obtained by averaging over 100 samples. First, in order to check the validity of the simulation and the reconstruction procedure, we perform the simulations for the scintillation-less case (the amplitude fluctuations **in the pupil plane** are set to zero). The results are plotted in Figure 4 by squares, and show that turbulence-induced phase distortions are retrieved with a suitable error (about 15%–20%). However, in the presence of scintillations the results are quite different (they are plotted Figures 4 and 5 by points, while some samples of intrafocal turbulence-degraded images are shown in Figure 6). Analyzing Figures 4 and 5 one can arrive at the same conclusion as in the case of non-correlated distortions shown in Figure 3: the Hickson scheme cannot provide a proper phase reconstruction in the presence of turbulence-induced amplitude fluctuations. Note that under closed-loop conditions the results may be different, but this study is beyond the scope of this paper.

## 4. CONCLUSIONS

We have compared the quality of phase retrieval in the case of non-uniform illumination for two curvature schemes: with two symmetrically-defocused images (Roddier scheme) and with a single defocused image (Hickson scheme). The results obtained have shown that the Roddier scheme allows a good phase reconstruction even under a quite strong level of amplitude fluctuations. This property of the Roddier scheme is due to the differential character of the scheme, which effectively suppresses the influence of amplitude fluctuations in the curvature signal. In contrast to the Roddier scheme, the scheme with a single defocused image is very sensitive to amplitude fluctuations: even a quite low level of them makes the phase reconstruction practically impossible.

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