

A NEW ANALYTIC APPROXIMATION TO THE ISOTHERMAL, SELF-GRAVITATING SPHERE

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RESUMEN

Derivamos una nueva prescripción analítica para la densidad de la esfera autogravitante, isotérmica no singular. La aproximación propuesta proporciona la estratificación de densidad con un error máximo de $\approx 1\%$. Esto es una mejora con respecto a prescripciones previas, que sólo daban este tipo de precisión sobre dominios radiales limitados.

ABSTRACT

We derive a new analytic prescription for the density of the non-singular, isothermal, self-gravitating sphere. The proposed prescription gives the density stratification for all radii with a maximum error of $\approx 1\%$. This is an improvement over previous prescriptions, which only gave such an accuracy over limited radial domains.

Key Words: galaxies: halos — ISM: clouds — stars: formation

1. INTRODUCTION

The solutions to the self-gravitating, isothermal sphere are relevant in two main contexts:

1. The equilibria of “molecular cloud cores” in star formation regions,
2. The equilibria of N-body systems.

An example of the first point is the work of Shu (1977), who studied the gravitational collapse of a singular, isothermal sphere, and of the second point is the work of Shapiro, Iliev, & Raga (1999), who modeled post-collapse, cosmological structures as non-singular isothermal spheres.

The lack of a full analytic solution for the non-singular isothermal sphere has led to the derivation of approximate, analytic forms that agree with the exact solution (obtained by integrating numerically the isothermal, Lane-Emden equation) to different degrees of accuracy. Two relatively recent papers on this topic are the ones of Liu (1996) and Natarajan & Lynden-Bell (1997). This type of approach is described in § 2.

In the present paper, we attempt to derive an analytic prescription that gives the density of the full, non-singular, isothermal sphere to within an accuracy of $\sim 1\%$. This accuracy is probably acceptable for the purpose of initializing numerical simulations which require such a structure as an initial condition.

To this effect, we present a new derivation of the “far field”, analytic solution for the isothermal, Lane-Emden equation (§ 3). This solution, dating back to the work of Emden at the turn of the 19th century, has apparently been forgotten in the more recent literature. We then derive a far field interpolation formula based on this analytic solution.

This “far field” interpolation is combined with a second, “near field” interpolation (described in § 4), to obtain a full analytic prescription for the non-singular density stratification. The errors associated with our proposed interpolation are compared to the ones of the work of Liu (1996) and Natarajan & Bell (1997) in § 5. A second possibility for obtaining the full, non-singular density stratification is to combine the approximation of Hunter (2001) with the analytic “far field” solution. This is described in § 6. In § 7, we present a discussion of the relative errors in the density gradient of our “full, approximate solutions”. Finally, a summary of the work is presented in § 8.

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2. THE ISOTHERMAL SPHERE

The hydrostatic equation for a self-gravitating, gaseous sphere is:

$$\frac{dP}{dR} = -\frac{\rho GM_R}{R^2}, \quad (1)$$

where P and ρ are the gas pressure and density (respectively), R is the spherical radius, and

$$M_R = 4\pi \int_0^R \rho R'^2 dR', \quad (2)$$

is the mass within a radius R . Setting $P = \rho c_0^2$ (where c_0 is the isothermal sound speed, assumed to be uniform), grouping terms and taking a derivative with respect to R one obtains the isothermal, Lane-Emden equation:

$$\frac{d}{dR} \left(R^2 \frac{d \ln \rho}{dR} \right) = -\frac{4\pi G}{c_0^2} \rho R^2. \quad (3)$$

An exact solution to this equation is the ‘‘singular, isothermal sphere’’:

$$\rho_S(R) = \frac{c_0^2}{2\pi G} \frac{1}{R^2}. \quad (4)$$

However, the non-singular isothermal sphere (which also satisfies equation 3) does not have a full, analytic solution. It is possible to propose a Taylor series of the solution, and substituting it into equation (3) to obtain the coefficients of this series by matching terms with the same power of R . To second order in R , one obtains the well known solution:

$$\rho_2(R) = \rho_c \left[1 - \left(\frac{R}{R_c} \right)^2 \right], \quad (5)$$

where ρ_c is the central density and

$$R_c \equiv \sqrt{\frac{3c_0^2}{2\pi G \rho_c}} \quad (6)$$

is the so-called ‘‘core radius’’ of the non-singular sphere. In the literature the ‘‘core radius’’ has been defined as in equation (6), but with varying numerical constants. Throughout the present paper, we use R_c as defined in equation (6).

If one defines dimensionless variables $r = R/R_c$ and $\rho' = \rho/\rho_c$, the isothermal, Lane-Emden equation becomes:

$$\frac{d}{dr} \left(r^2 \frac{d \ln \rho'}{dr} \right) = -6\rho' r^2. \quad (7)$$

The dimensionless, singular solution then is:

$$\rho'_S(r) = \frac{1}{3r^2}, \quad (8)$$

and the second order, $r \ll 1$, solution is:

$$\rho'_2(r) = 1 - r^2. \quad (9)$$

It is a well-known result that the non-singular solution (obtained, e.g., by integrating numerically equation 7 with the boundary conditions $\rho' = 1$ and $d\rho'/dr = 0$ at $r = 0$) asymptotically approaches the singular solution (equation 8) for $r \gg 1$. A certain amount of effort has been made to design interpolation formulae that match the Taylor series expansion (equation 9) for small r and the singular solution (equation 8) for large r , and reproduce the non-singular solution (for all r) with a certain degree of accuracy.

The simplest of the proposed interpolations is:

$$\rho'_a(r) = \frac{1}{1+r^2}. \quad (10)$$

This interpolation has the virtue of coinciding (to second order in r) with the Taylor series solution (equation 9), and having a $\rho \propto r^{-2}$ dependence for $r \gg 1$. However, the proportionality constant ($= 1$) of the large r dependence of equation (10) is larger by a factor 3 than the constant of the singular solution (see equation 8).

A possibility is to propose Taylor series expansions for $\rho'(r)$ or for $1/\rho'(r)$ to higher orders in r . For example, the substitution of an 8th-order expansion of $1/\rho'$ into the isothermal Lane-Emden equation (equation 7) gives:

$$\rho'_8(r) = \left[1 + r^2 + \frac{1}{5}r^4 - \frac{2}{105}r^6 + \frac{4}{1575}r^8 \right]^{-1}. \quad (11)$$

Unfortunately, this type of expansion converges to the ‘‘exact’’ (i.e., numerical) non-singular isothermal sphere solution only for $r < 1$.

Another possibility is to study Taylor series expansions centered at a radius of order 1 (rather than around $r = 0$, as above). This type of expansion has been studied for the Lane-Emden equation with different polytropic indices (though not for the isothermal case) by Mohan & Al-Bayaty (1980). Other ways of obtaining convergent series have been explored by Nouh (2004), who does consider the isothermal case.

Yet another possibility is to propose interpolations with different functional forms. For example,

Natarajan & Lynden-Bell (1997) used an interpolation formula with two additive terms of the general form of the right hand side of equation (10). Their interpolation formula gives reasonably accurate results (to within $\sim 1\%$ from the “exact” solution) for $R < 4 R_c$, but diverges from the correct solution for larger radii. The difference between the interpolation formula and the exact solution reaches values of $\sim 20\%$ at $R \sim 100 R_c$ (with R_c given by equation 6) and then slowly diminishes for larger radii (a result of the fact that the interpolation formula has the correct asymptotic behaviour for $R \rightarrow \infty$).

Another, more complex interpolation formula has been proposed by Liu (1996). This formula reproduces the exact solution to within $\sim 1\%$ for $R < 60 R_c$, then reaches a maximum deviation of $\sim 6\%$ at $R \approx 150 R_c$, and finally converges slowly to the exact solution at larger radii. We should note that both Liu (1996) and Natarajan & Lynden-Bell (1997) define a core radius a factor of $\sqrt{6}$ smaller than the one of our equation (6).

In the following section, we address the issue of why the previous interpolation formulae have such difficulties to reproduce the exact solution at radii $R \sim 100 R_c$, and propose a possible way of fixing the problem.

3. THE $R \gg R_c$, “FAR FIELD” SOLUTION

For $R \gg R_c$ (i.e., for $r \gg 1$), the non-singular solution of the isothermal, Lane-Emden equation (equation 7) asymptotically approaches the singular solution (equation 8). In order to study the characteristics of this approach, we propose a function $q(r)$ such that:

$$\rho'(r) = [1 + q(r)] \rho'_s(r), \quad (12)$$

where $\rho'_s(r) = 1/(3r^2)$ (see equation 8). Substituting equation (12) into equation (7) we obtain a differential equation for q of the form:

$$2q + \frac{2rq'}{1+q} - \frac{r^2q'^2}{(1+q)^2} + \frac{r^2q''}{1+q} = 0, \quad (13)$$

where $q' = dq/dr$ and $q'' = d^2q/dr^2$. We now assume that $q, q', q'' \ll 1$, and therefore only keep in equation (13) the terms with linear dependencies on these functions. We then obtain:

$$2q + 2rq' + r^2q'' = 0. \quad (14)$$

Proposing a solution of the form $q = Cr^p$ (with constant C and p) and substituting into equation (14), one straightforwardly obtains that $p =$

$1/2 + i\sqrt{7}/2$. Therefore, the real (as opposed to complex) solution has the form:

$$q_l(r) = \frac{A}{r^{1/2}} \cos\left(\frac{\sqrt{7}}{2} \ln r + \phi\right), \quad (15)$$

in which A and ϕ are the two integration constants. This solution is derived in a more elaborate way in the book of Chandrasekhar (1967, first published in 1939).

From equation (15), we see that indeed $q_l(r) \rightarrow 0$ (in other words, $\rho' \rightarrow \rho'_s$, see equation 12) for $r \gg 1$, but that it changes sign periodically as a function of $\ln r$. Therefore, the non-singular solution crosses the singular solution repeated times, with intervening excursions of decreasing amplitude for larger values of r (see equation 15).

We have now taken the “exact” solution $\rho_{\text{ex}}(r)$ of the isothermal, Lane-Emden equation (obtained by integrating equation 7 with a second-order, variable step method), and used it to compute the corresponding deviation q_{ex} from the singular solution:

$$q_{\text{ex}}(r) = \frac{\rho_{\text{ex}}(r)}{\rho_s(r)} - 1, \quad (16)$$

as follows from equation (12). We then obtain the values $A_0 = 0.735$ and $\phi_0 = 5.396$ from a least squares fit of $q_l(r)$ (equation 15) to the exact solution (equation 16) in the $r = 10 \rightarrow 10^4$ radial range. The deviations of the exact solution $q_{\text{ex}}(r)$ and the linearized solution $q_l(r)$ (equation 15, with integration constants A_0 and ϕ_0) from the isothermal solution are shown in Figure 1.

From this figure, we see that for $\log_{10} r > 1.5$ $q_l(r)$ closely follows the exact, $q_{\text{ex}}(r)$ solution. For smaller radii, appreciable differences appear, and the exact and linear solutions differ from each other by $\sim 10\%$ at $\log_{10} r = 0.2$.

In order to have an interpolation formula for $q(r)$ that has better accuracy at smaller values of r , we allow A and ϕ (see equation 15) to be functions of r of the form:

$$A = 0.735 \left(1 + \frac{a}{r^\alpha}\right), \quad \phi = 5.396 \left(1 + \frac{b}{r^\beta}\right), \quad (17)$$

and from a least squares fit to $\rho_{\text{ex}}(r)$ in the $r = 2 \rightarrow 100$ range we obtain $a = 5.08$, $\alpha = 1.94$, $b = 0.92$ and $\beta = 2.31$. The resulting formula q_{far} for the deviation from the singular solution then takes the form:

$$q_{\text{far}}(r) = \frac{0.735 (1 + 5.08r^{-1.94})}{r^{1/2}} \times \cos\left[\frac{\sqrt{7}}{2} \ln r + 5.396 (1 + 0.92r^{-2.31})\right]. \quad (18)$$

This approximation for $q(r)$ (see equation 12) is shown in Figure 1, where we can see that it follows the exact, $q_{\text{ex}}(r)$ solution down to $\log_{10} r \approx 0.3$.

The “far field” density distribution associated with $q_{\text{far}}(r)$ can straightforwardly be computed as:

$$\rho'_{\text{far}}(r) = \frac{1 + q_{\text{far}}(r)}{3r^2}, \quad (19)$$

following the definition of $q(r)$ (see equation 12).

We end this section with a short summary:

- One proposes a solution of the form $\rho'(r) = [1 + q(r)]\rho'_s(r)$, where $\rho'_s(r)$ is the singular solution.
- Substituting into Lane-Emden’s equation, and keeping only linear terms in q and its derivatives, one obtains a differential equation with an analytic solution $q_l(r)$ (see equation 15). This solution reproduces very well the $r > 30$ region of the exact (i.e., numerical) solution of the isothermal, Lane-Emden equation (see Figure 1).
- Inspired by the form of $q_l(r)$, we generate a more complex formula in which both A and ϕ (constants in the linear solution) are allowed to be chosen functions of r . The additional constants are then obtained through a fit to the exact solution.

In this way, we obtain an interpolation formula $\rho_{\text{far}}(r)$ (see equations 18 and 19) which reproduces satisfactorily the exact solution for $r > 2$ ($\log_{10} r > 0.3$, see above). The errors in this approximate solution are quantified in § 5.

4. THE “SMALL R” SOLUTION

In order to obtain the full density stratification of the isothermal sphere, the interpolation formulae derived in the previous section (equations 18 and 19, which are valid for $R > 2 R_c$) have to be extended with an internal solution (covering the $R < 2 R_c$ region). To this effect, one can choose the interpolation formula of Natarajan & Lynden-Bell (1997):

$$\rho'_{\text{nl}} = \frac{A}{1 + 6r^2/a} - \frac{B}{1 + 6r^2/b}. \quad (20)$$

These authors propose two main possibilities:

- approximation NL_a : $A = 5$, $a = 10$, $B = 4$ and $b = 12$. With these values, equation (20) gives the correct behaviour to second order in r for $r \ll 1$, and has the appropriate limit, coinciding with the singular solution for $r \rightarrow \infty$,

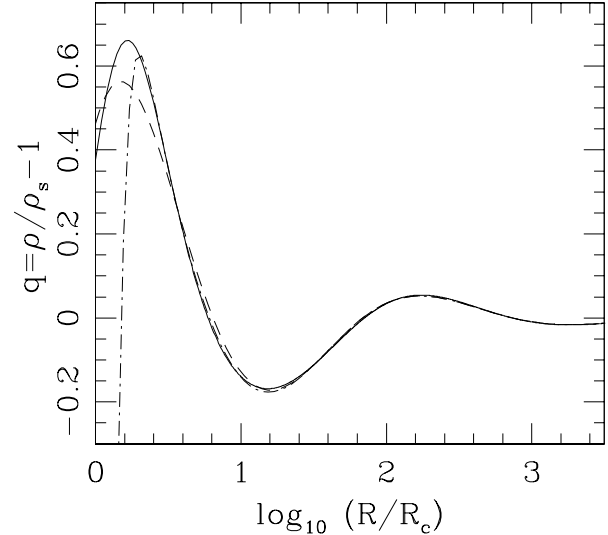


Fig. 1. Deviations $q = \rho/\rho_s - 1$ from the singular solution as a function of r . We show the deviations of the “exact” solution $q_{\text{ex}}(r)$ (solid line), of the linearized solution $q_l(r)$ (dashes) and the “far field” interpolation $q_{\text{far}}(r)$ (dot-dash).

- approximation NL_b : $A = 4.365$, $a = 10.157$, $B = 3.365$ and $b = 12.664$. These coefficients result in a fit which does not satisfy asymptotically the $r \rightarrow 0$ and $r \rightarrow \infty$ behaviours, but which agrees well with the exact solution out to larger values of r .

The relative errors

$$\epsilon(r) = \frac{\rho'(r)}{\rho_{\text{ex}}(r)} - 1 \quad (21)$$

with respect to the “exact” solution $\rho_{\text{ex}}(r)$ of these two approximations of Natarajan & Lynden-Bell (1997) are shown in the top frame of Figure 2.

We propose an alternative interpolation formula of the form:

$$\rho'_{\text{near}}(r) = \frac{1 + (2c - 1)r^2}{(1 + cr^2)^2}. \quad (22)$$

This formula coincides with the second-order, $r \ll 1$ solution (equation 9). If one chooses $c = 3 - \sqrt{6} \approx 0.551$, the interpolation formula also coincides with the singular solution for $r \rightarrow \infty$.

In the bottom frame of Figure 2, we show the errors associated with the interpolation formula given by equation (22) for $c = 0.551$ (i.e., the value that results in the singular solution for $r \rightarrow \infty$), $c = 0.548$ (obtained from a least-squares fit to the exact solution in the $r = 0 \rightarrow 2$ range), and $c = 0.546$ (giving a better fit at somewhat larger values of r).

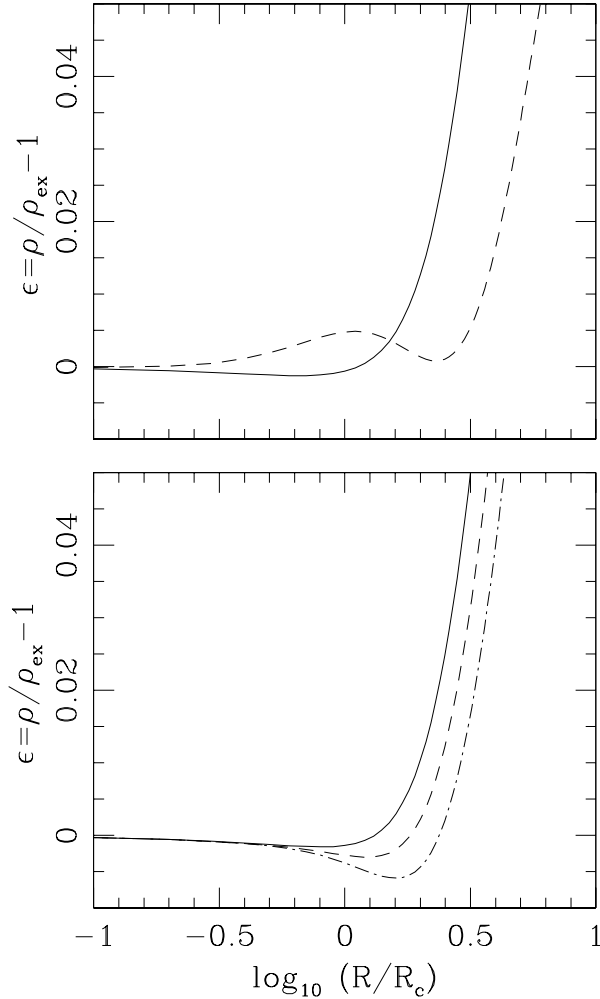


Fig. 2. Fractional deviations $\epsilon = \rho/\rho_{\text{ex}} - 1$ of the analytic approximations with respect to the “exact” (i.e., numerical) solution of the non-singular, isothermal sphere. The top frame shows the $\epsilon(r)$ dependencies obtained with the two interpolations proposed by Natarajan & Lynden-Bell (1997): NL_a (solid line) and NL_b (dashed line), described in § 4. The bottom frame shows the $\epsilon(r)$ dependencies resulting from our proposed interpolation (equation 22) with $c = 0.551$ (solid line), 0.548 (dashes) and 0.546 (dot-dash).

In the following, we adopt the $c = 0.548$ solution as the “near field” (i.e., low r) interpolation for the non-singular density stratification. This solution crosses our “far field” interpolation (equation 19) at $r_0 = 2.312$. Therefore, for describing the whole, non-singular density stratification we use equation (22) for $r \leq r_0$ and equation (19) for $r > r_0$. The errors associated with this choice are discussed in the following section.

5. THE FULL SOLUTION

In order to obtain an approximation to the non-singular solution for all radii, we use equation (22, with $c = 0.548$) for $r \leq r_0 = 2.312$, and equations (18–19) for $r > r_0$. In Figure 3, we show the resulting $\rho'(r)$ density stratification, and its associated fractional deviation $\epsilon(r)$ from the exact solution $\rho'_{\text{ex}}(r)$ (defined in equation 21). From this figure, we see that our full interpolation has a maximum deviation of ≈ -0.0101 at $\log_{10} r \approx 1.2$ ($r \approx 16$, central frame).

In Figure 3, we also show the results obtained from approximation “ NL_a ” (of Natarajan & Lynden-Bell 1997, see § 4), which shows good agreement with the exact solution only for r smaller than ~ 1 , and reaches a maximum fractional deviation of ≈ 0.24 at $r \approx 13$ (this peak in the deviation lies off the plot in the central frame of Figure 3). Qualitatively similar results are obtained with approximation “ NL_b ” (of Natarajan & Lynden-Bell 1997, see § 4) and with our “near field” interpolation (equation 22), which is not shown in Figure 3. Therefore, in order to obtain a reasonably accurate description of the $r > 1$ density stratification, it is necessary to switch (as was done above) from these interpolations to a complementary “far field” interpolation.

Such a switch between two interpolations was done by Liu (1996). The fractional deviation between his proposed interpolation and the exact solution is also shown in the bottom frame of Figure 3. It is clear that for $\log_{10} r < 1.7$, Liu’s interpolation has associated errors which are similar to the ones of our new interpolation formula. However, for larger radii it has a maximum (negative) deviation of ≈ -0.058 at $\log_{10} r \approx 2.2$, followed by a positive peak in the deviation of ≈ 0.016 at $\log_{10} r \approx 3.26$. This oscillation of the error at large radii is a result of the fact that the “far field” interpolation formula of Liu (1996) models the first negative excursion of $q(r)$ (at $\log_{10} r \sim 1.2$, see Figure 1), but does not have the periodic behaviour of $g(r)$ as a function of $\ln r$ of the exact solution at large radii (which is reproduced by the $q \ll 1$ analytic solution described in § 2).

Therefore, our combination of near/far field solutions have a maximum error ($\approx 1\%$, see above) which is approximately a factor of 6 lower than the one of Liu’s (1996) approximation (which has a maximum deviation of $\approx 6\%$ from the exact solution, see above).

6. ALTERNATIVE, FULL SOLUTION

Also shown in Figure 3 (bottom frame) is the approximation of Hunter (2001), who extended the

interpolation formula of Natarajan & Lynden-Bell (1997) to a four-term sum:

$$\rho_{\text{H}}(r) = \sum_{j=1}^4 \frac{A_j}{a_j^2 + 6r^2}, \quad (23)$$

where the A_j and a_j^2 coefficients are given in Table 2 of Hunter (2001). This interpolation has a good behaviour out to $\log_{10} r \approx 1.5$, and then follows Liu’s (1996) approximation at larger radii (therefore also showing a maximum deviation of $\approx 6\%$ from the exact solution).

An alternative approximation to the full non-singular solution is to use equation (23) for $r \leq r_1 = 27.643$, and the “linear q ” solution (equation 15) with the appropriate phase and amplitude for $r > r_1$. Therefore, the “far field” $r > r_1$ interpolation for the density has the form:

$$\rho'_{\text{far},2} = \frac{1 + 0.735r^{-1/2} \cos[(\sqrt{7}/2) \ln r + 5.396]}{3r^2}, \quad (24)$$

The interpolation given by equations (23–24) has a maximum error of $\approx 0.4\%$ with respect to the exact solution (see the bottom frame of Figure 3).

7. THE ERRORS IN THE PRESSURE FORCE

As discussed in § 1, one of the principal purposes of having approximate non-singular isothermal sphere solutions is to use them to initialize numerical simulations of self-gravitating flows. Such simulations include pressure gradient forces in the momentum and energy equations of the form $dP/dR = c_0^2 d\rho/dR$.

It is therefore interesting to evaluate the fractional error in the density gradient:

$$\epsilon_d(r) = \frac{d\rho'/dr}{d\rho_{\text{ex}}/dr} - 1 = \frac{d\rho'}{d\rho_{\text{ex}}} - 1, \quad (25)$$

where $\rho'(r)$ is the approximate solution and ρ_{ex} the “exact” (i.e., numerical) solution of the Lane-Emden equation.

The $\epsilon_d(r)$ dependencies obtained from our two proposed “full solutions” (described in §§ 5 and 6, respectively) are shown in Figure 4. For $r' > 1$ ($R > R_c$), the solution of § 5 has a maximum fractional error $|\epsilon_{d,m}| \approx 0.025$, and the solution of § 6 has $|\epsilon_{d,m}| \approx 0.01$. In both solutions, the maximum fractional error (for $r' > 1$) occurs at the radii at which we have the switch from the “small R ” to the “far field” solutions.

It would be possible to remove the discontinuity in the pressure force (which reflects the behaviour of

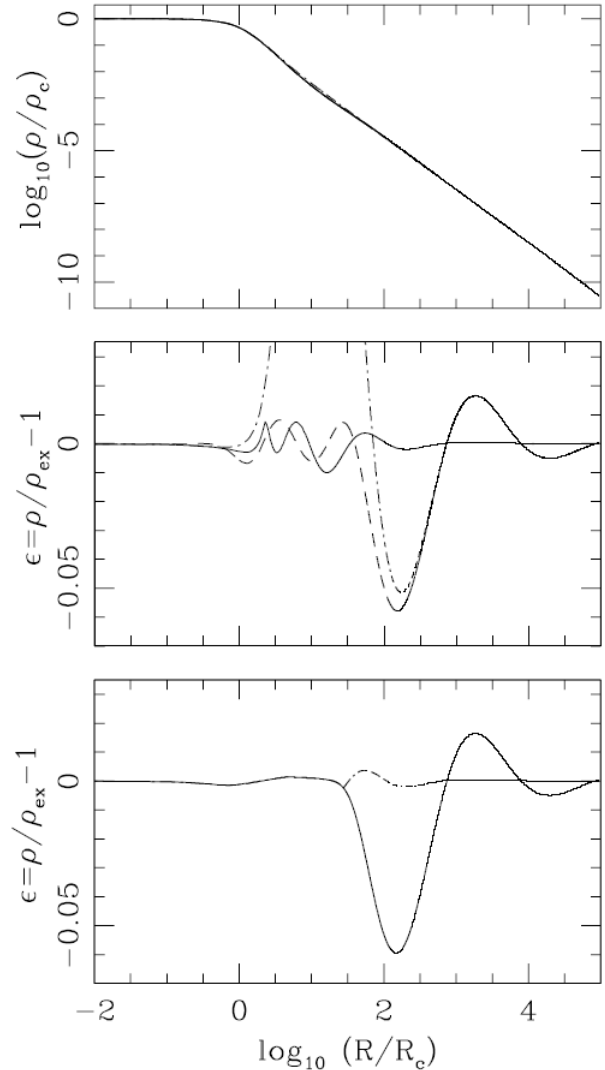


Fig. 3. Top frame: “exact” (i.e., numerical), non-singular isothermal sphere density stratification (solid line), and the stratifications obtained from the different approximations discussed in this paper (which are basically indistinguishable from the exact solution at the scale of this graph). Central frame: fractional errors $\epsilon = \rho/\rho_{\text{ex}} - 1$ with respect to the “exact” solution of the new interpolation described in §§ 3 and 5 (solid line), of Liu’s (1996) interpolation (dashes) and of the NL_a interpolation (dot-dash) of Natarajan & Lynden-Bell (1997) described in § 4 (the NL_b interpolation is not shown). Bottom frame: Hunter’s (2001) interpolation (solid line) and the combination of this approximation with our analytic “far field” solution (dashed line), as described in § 6.

$d\rho'/dr$) at the matching points (between the small R and the far field solutions) by using a floating average between the two solutions in a limited region around the matching points. However, as can be

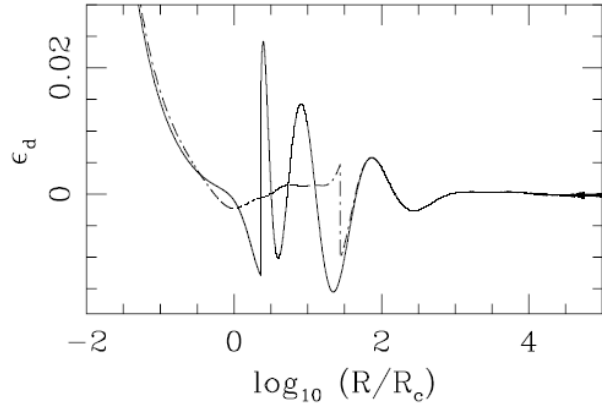


Fig. 4. Fractional errors ϵ_d (in $d\rho/dR$, see equation 25) as a function of $r = R/R_c$ for the “full” solutions described in § 5 (solid curve) and § 6 (dashed curve).

seen from Figure 4, this will not reduce the errors in the pressure force in a dramatic way.

Interestingly, both solutions show a drastic increase in the relative error (see equation 25) with decreasing r in the $r < 0.1$ region. This is unlikely to be a problem since the pressure force has very low values in this region.

8. SUMMARY

We have obtained two combinations “near” and “far field” interpolation formulae (§§ 5 and 6) that give analytic approximations for the full, non-singular, self-gravitating isothermal sphere with maximum deviations of 1% and 0.6% (respectively) from the exact solution. This accuracy is probably sufficient for the purpose of initializing either gasdynamic or N-body simulations which require an equilibrium, isothermal, self-gravitating structure as an initial condition.

The previous interpolations proposed by Liu (1996) and by Hunter (2001) have a maximum deviation of $\approx 6\%$ from the exact solution. The improvement of accuracy in our solution is due to the fact that we explicitly use the “far field, linear solution” of the Lane-Emden equation (in which it is assumed that the fractional deviation q between the exact so-

lution and the singular sphere is much less than 1). Because of this, our proposed interpolation shows an improved convergence to the exact solution at large radii.

We present an apparently new and almost trivial derivation of the “far field solution” (§ 3). Chandrasekhar (1967) presents a considerably more complex derivation, which he attributes to Emden in a detailed discussion of credits at the end of Chapter IV.

It is clear that the analytic “far field solution” offers many possibilities for doing extensions to obtain a full description of the non-singular solution. Even though obtaining interpolations to better accuracies (than the one of the presently proposed interpolation) is probably not worth a large amount of effort, it might be interesting to try to find simpler forms. Such a pursuit might be worthwhile for researchers who enjoy playing such games. There are of course other methods for approximating the non-singular solution which should also be developed further (see Roxburgh & Stockman 1999 and Mirza 2009).

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