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A NEW APPROXIMATE ANALYTIC SOLUTION FOR THE IONIZATION STRUCTURE OF A STRÖMGREN SPHERE

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RESUMEN

(Strömgren 1939) presentó un modelo para la fotoionización de una nebulosa homogénea, y derivó soluciones analíticas aproximadas de su modelo. El presente artículo explora el modelo de Strömgren y muestra que es posible construir soluciones analíticas más precisas (aunque por supuesto más complejas). Estas soluciones son interesantes para inicializar simulaciones numéricas y para tener una descripción analítica adecuada de regiones fotoionizadas con zonas transición HII→HI "gruesas".

ABSTRACT

(Strömgren 1939) presented a model for the photoionization of a homogeneous nebula, and derived approximate analytic solutions to his model. The present paper explores Strömgren's model, and shows that it is possible to construct more accurate (though of course more complex) analytic solutions. These new solutions are interesting for initializing numerical simulations and for obtaining an appropriate analytic description of photoionized regions with "thick" HII→HI transition regions.

Key Words: ISM: HII regions

1. INTRODUCTION

(Strömgren 1939) first derived a "grey" (i.e., frequency independent) radiative transfer+photoionization equilibrium model for a photoionized region with uniform density and temperature. He derived two analytic solutions:

1. an "inner solution" for the ionization stratification within the ionized region with the assumption that $x = n_{HII}/n_H \approx 1$ (where n_{HII} is the ionized H and n_H the total H density),
2. a solution to the HII→HI transition region under the assumption that this region is thin compared to the outer radius of the nebula (so that the geometrical dilution of the stellar UV photons can be assumed to be constant across the transition region).

The first solution gives a correct description for the internal, almost fully ionized part of an HII region, while the second solution provides an adequate description of the transition region, provided that it is indeed thin compared to the so-called "Strömgren radius" (R_S).

The present paper discusses a new, approximate analytic solution to Strömgren's problem, which can be used for photoionized regions with either thin or thick HII→HI transition regions. Strömgren's "inner solution" is rederived in § 2, in a form appropriate for an extension to more accurate, approximate analytic solutions. An example of such a solution is presented in § 3, and a comparison with the exact (i.e., numerical) solutions is shown in § 4. Finally, a discussion of the relevance of the new solution is presented in § 5.

2. STRÖMGREN'S ANALYSIS

In this section, we basically rederive the solution obtained by (Strömgren 1939). We consider an isothermal, constant density pure H nebula photoionized by a point source producing S_* ionizing photons per unit time. The ionization equilibrium resulting from the balance between photoionizations and recombinations in an arbitrary point of the nebula can be written as:

$$n_H(1-x)\phi_H = x^2 n_H^2 \alpha_H \,, \tag{1}$$

where $x = n_{HII}/n_H = n_e/n_H$ is the ionization fraction (with n_H being the uniform H number density, and n_{HII} and n_e being the ionized H and electron densities, respectively), α_H the recombination coefficient and ϕ_H the photoionization rate of H.

This quadratic equation for x can be inverted to obtain:

$$x = \frac{1}{2A}\left(\sqrt{1+4A}-1\right)\,, \tag{2}$$

with

$$A \equiv \frac{n_H \alpha_H}{\phi_H} = \frac{1-x}{x^2}\,. \tag{3}$$

The H photoionization rate is given by

$$\phi_H = \int_{\nu_0}^{\infty} \frac{4\pi J\nu}{h\nu}\sigma_\nu \, d\nu = \frac{S_* \sigma_H}{4\pi R^2} e^{-\tau}\,, \tag{4}$$

with

$$\tau = n_H \sigma_H \int_0^R (1-x) dR\,, \tag{5}$$

where J_ν is the angular average of the specific intensity, σ_ν is the (frequency dependent) H photoionization cross section, ν_0 is the Lyman limit frequency and R is the spherical radius. The second equality in equation (4) is obtained in the "grey approximation of the ISM" (i.e., setting $\sigma_\nu = \sigma_H$ independent of ν), considering the central star as a point source, and neglecting the contribution of the diffuse ionizing photon field.

Following (Strömgren 1939), we now use equations (3) and (4) to define

$$f \equiv e^{-\tau} = \frac{3r^2}{A\lambda} \tag{6}$$

in terms of the dimensionless radius

$$r = \frac{R}{R_S} \quad \text{with } R_S \equiv \left(\frac{3S_*}{4\pi n_H^2 \alpha_H}\right)^{1/3}\,, \tag{7}$$

and where the dimensionless parameter

$$\lambda \equiv R_S n_H \sigma_H \tag{8}$$

is the ratio between the Strömgren radius R_S and the mean free path of the ionizing photons in the neutral gas.

Now, from equation (5) we see that f obeys the differential equation

$$\frac{1}{f}\frac{df}{dr} = -\lambda(1-x)\,, \tag{9}$$

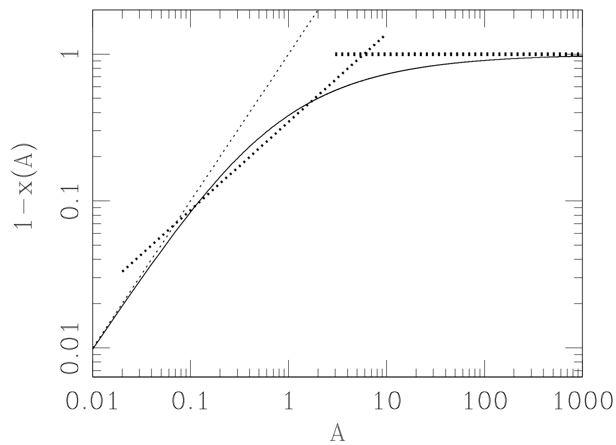

Fig. 1. The fraction $1 - x(A)$ of neutral H (where $x = n_{HII}/n_H$ is the HII fraction) as a function of the A parameter (see equation 3). The exact solution (equation 2) is shown with the solid curve. The linear approximation to $1-x(A)$ (equations 10 and 14) is shown with the thin, dotted line. The other two segments of the three-power law approximation (equation 14) are shown with thicker dotted lines.

where x is given as a function of A by equation (2). This differential equation can be integrated numerically with the boundary condition $f(0) = 1$, but no exact analytic integral has been found.

In order to obtain an approximate numerical integral, one expands the term in parentheses on the right hand side of equation (2) to second order in A, obtaining:

$$1 - x(A) \approx A\,. \tag{10}$$

This Taylor series expansion is compared with the exact form of $1 - x(A)$ (obtained from equation 2) in Figure 1.

We now write A in terms of f and r using equation (6) and insert the result into equations (10) and (9) obtaining the differential equation

$$\frac{df_S}{dr} = -3r^2\,, \tag{11}$$

which can be directly integrated with the boundary condition $f_S(0) = 1$ to obtain the approximate solution

$$f_S(r) = 1 - r^3\,. \tag{12}$$

If one now combines this result with equations (6) and (2) one finally obtains an approximate form for the ionization fraction as a function of dimensionless radius:

$$x_S(r) = \frac{\lambda(1-r^3)}{6r^2}\left[\sqrt{1+\frac{12r^2}{\lambda(1-r^3)}}-1\right]\,. \tag{13}$$

This solution is compared with an "exact" numerical integration of equations (9) and (2) for three values of the dimensionless parameter $\lambda = 10$, 100 and 1000 in Figure 2. Regardless of the value of λ, $x_S(r)$ goes to zero at $r = 1$ (i.e., at $R = R_S$, the Strömgren radius).

The approximate solution $f_S(r)$ (equation 12) for the structure of a photoionized sphere was obtained by (Strömgren 1939), who noted that inside an HII region we can set $x \approx 1$ in the right hand side of equation (1), leading to the approximate differential equation (11). (Strömgren 1939) then proceeded to find a second approximate solution by setting $R = R_S$ in the denominator of the right hand side of equation (4) and then deriving an approximate form of $x(r)$ for radii around the Strömgren radius. This solution is appropriate for the thin transition between ionized and neutral regions that occurs for $\lambda \gg 1$, but is not completely satisfactory because the natural boundary condition at the origin (i.e., $f(0) = 1$) cannot be applied (see, e.g., the discussion in the book of Dyson & Williams 1980).

3. NEW APPROXIMATE SOLUTION

In the derivation of Strömgren's solution given in § 2, we show the way to obtain more accurate analytic approximations to the structure of a photoionized sphere. We have the differential equation (9) with the right hand side $1 - x(A)$ given by equation (2). Then, using equation (6) we substitute $A = 3r^2/(\lambda f)$ and obtain a differential equation with f and r as variables, which has to be integrated to obtain $f(r)$ (with the boundary condition $f(0) = 1$). Unfortunately, given the complex form of $x(A)$ (see equation 2), the resulting differential equation does not have an analytic integral.

A process that can be used to find approximate solutions to the problem is to replace $1-x(A)$ (on the right hand side of equation 9) by approximate forms which do lead to an analytically integrable differential equation. In § 2 we set $1 - x(A) \approx A$ (corresponding to the first order Taylor series expansion of equation 10), leading to the approximate analytic solution of (Strömgren 1939).

We now propose the following three-segment approximation for the $1 - x(A)$ function:

$$
\begin{aligned}
1 - x(A) &= A\,; \quad A < A_1 \\
&= aA^b\,; \quad A_1 \le A \le A_2 \\
&= 1\,; \quad A > A_2\,,
\end{aligned}
\tag{14}
$$

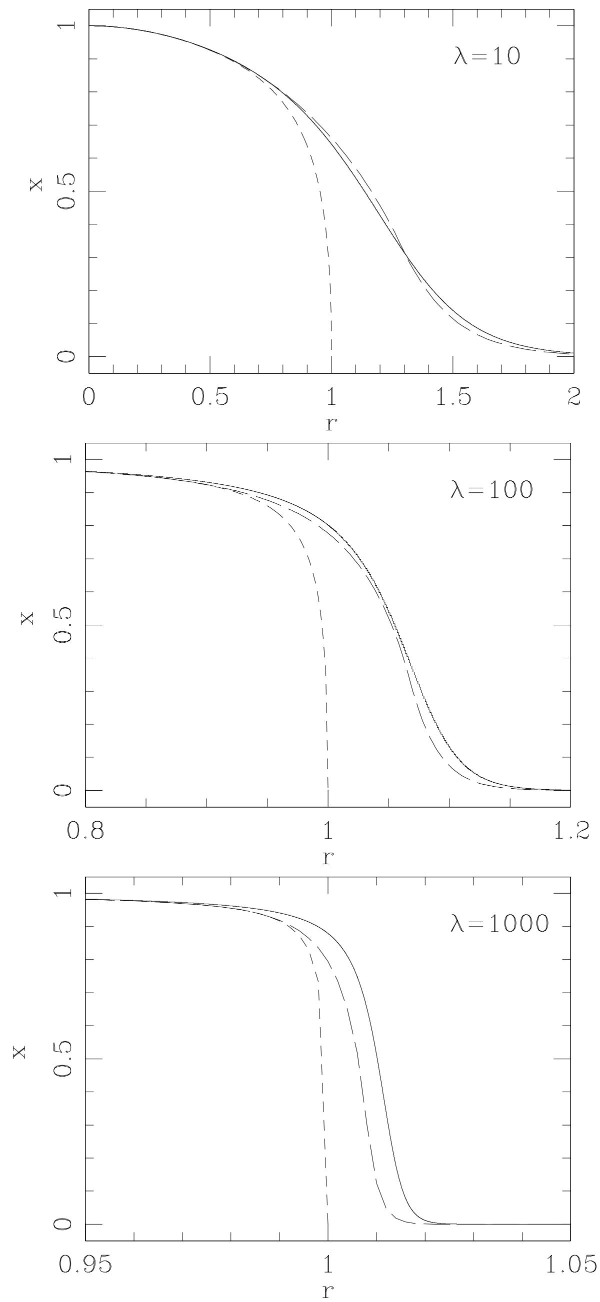

Fig. 2. The ionization fraction $x = n_{HII}/n_H$ as a function of dimensionless radius $r = R/R+S$ (where R_S is the Strömgren radius, see equation 7) for three values of the dimensionless parameter (see equation 8) $\lambda = 10$ (top), $\lambda = 100$ (center) and $\lambda = 1000$ (bottom). Note that in the central and bottom frames the graphs do not extend to the position of the central star (i.e., to $r = 0$). The solid curves correspond to the exact (i.e., numerical) solution, the short dash curves to Strömgren's solution (see § 2) and the long dash curves to the new, three-segment approximate analytic solution (see § 3).

with $A_1 = 0.07$, $A_2 = 5.887$, $a = 0.345$ and $b = 0.600$. This approximation is compared with the value of $1 - x(A)$ obtained using the exact form for $x(A)$ (given by equation 2) in Figure 1.

With the approximate form for $1-x(A)$ given by equation (14), equation (9) can be integrated analytically to obtain:

$$f(r) = 1 - r^3 \,, \tag{15}$$

for $r \le r_1$,

$$f(r) = \left[f_1^b - \frac{3^b a b \lambda^{1-b}}{2b+1} \left(r^{2b+1} - r_1^{2b+1} \right) \right]^{1/b} \,, \tag{16}$$

for $r_1 < r \le r_2$, with $f_1 = 1 - r_1^3$,

$$f(r) = f_2 e^{\lambda(r_2 - r)} \,, \tag{17}$$

for $r_2 < r$. In equations (15-16) the a and b constants have the values given by equation (14). In equation (17), f_2 is the value given by equation (16) when evaluated in $r = r_2$.

The switch between equations (15) and (16) occurs at a dimensionless radius

$$r_1 = \left(1 + \frac{1}{\lambda A_1} \right)^{-1} \,, \tag{18}$$

where the value of A_1 is given in the text following equation (14). This relation is obtained from a second-order Taylor series expansion around $r_1 = 1$.

The switch between equations (16) and (17) occurs at a dimensionless radius r_2, which is the root of the equation:

$$p_1 r_2^{2b+1} + p_2 r_2^{2b} + p_3 = 0 \,, \tag{19}$$

with

$$p_1 = \frac{3^b a b \lambda^{1-b}}{2b+1} \,; \;\; p_2 = \frac{3^b a}{\lambda^b} \,;$$

$$p_3 = -f_1^b - \frac{3^b a b \lambda^{1-b}}{2b+1} r_1^{2b+1} \,. \tag{20}$$

An approximate, analytic expression for r_2 can be obtained by setting

$$r_2 = 1 + \eta \,, \tag{21}$$

and then expanding equation (19) to second order in η to obtain:

$$\alpha \eta^2 + \beta \eta + \gamma = 0 \,, \tag{22}$$

with

$$\alpha = (2b+1) b p_1 + b(b-1) p_2 \,; \;\; \beta = (2b+1) p_1 + 2b p_2 \,;$$

$$\gamma = p_1 + p_2 + p_3 \,. \tag{23}$$

Using the "+ sign root" of equation (22), we can then obtain r_2 from equation (19).

4. RESULTS

We now take the 3-segment, approximate solution of equations (15-17) and substitute into equations (6) and (2) to obtain the ionization fraction x as a function of dimensionless radius r. The resulting form of $x(r)$ is compared with the exact (i.e., numerical) solution for values of the dimensionless parameter $\lambda = 10$, 100 and 1000 in Figure 2.

The analytic solutions clearly follow quite closely the exact solutions for the $\lambda = 10$ and 100 cases (top two panels of Figure 2). For the case with $\lambda = 1000$ (bottom frame of Figure 2), the analytic solution shows a $x = 1 \to 0$ transition with a spatial structure similar to the one of the exact solution, but with a spatial offset of ~ 0.005 in the dimensionless radial coordinate.

The errors in the $x(r)$ stratifications obtained with the approximate form for $1 - x(A)$ which we have chosen (see equation 14) are typical of well chosen 3-power law segment approximations. In order to obtain significantly more accurate analytic solutions for $x(r)$, it is necessary to go to 4-power law segment approximations for $1-x(A)$. It is, however, not clear that the increase in the complexity of the resulting analytic solutions is worthwhile.

5. DISCUSSION

Given the century-old flavor of this work, a discussion of its relevance appears to be appropriate. Firstly, the newly derived approximate analytical solution for the ionization structure of an HII region is satisfying in itself, in that it gives a more accurate description than the solution of Strömgren (1939). In the world of detailed numerical simulations of photoionized flows (see, e.g., Tremblin et al. 2012), such analytic solutions are useful both for initializing the simulations and for checking whether or not the simulations are giving accurate results.

For HII regions photoionized by stellar photon sources, the structure of the transition between ionized and neutral gas at the Strömgren radius is of course a rather minor point. This is because the dimensionless parameter λ (see equation 8) has values

$$\lambda = 1330 \left(\frac{S_*}{10^{49}\mathrm{s}^{-1}} \right)^{1/3} \left(\frac{n_H}{1\,\mathrm{cm}^{-3}} \right)^{1/3} \left(\frac{\sigma_H}{\sigma_{\nu_0}} \right) \,, \tag{24}$$

where we have used typical parameters for a galactic HII region. For a photoionizing spectrum emitted by stellar sources, the value of the frequency-independent σ_H that has to be used is $\sigma_H \approx \sigma_{\nu_0}$ (where $\sigma_{\nu_0} = 6.30 \times 10^{-18}\mathrm{cm}^2$ is the photoionization cross section of H at the Lyman limit). This is due to the fact that the stellar UV spectrum falls exponentially for frequencies close to the Lyman limit, so that the relevant absorption cross section σ_H (used in a grey model) has to be $\approx \sigma_{\nu_0}$ (as σ_ν falls only as ν^{-3}). For this case, the last term in parentheses on the right hand side of equation (24) is of order unity, and we then have $\lambda \sim 1000$.

Therefore, for a galactic HII region we have ionization fraction distributions similar to the $\lambda = 1000$ solution shown in the bottom panel of Figure 2. As noted by Strömgren (1939) and many papers in the intervening years, this result implies that the transition region (from fully ionized to neutral H) is very thin compared to the outer radius of the nebula. Because of this a "Strömgren sphere" description (in which the transition from HII to HI is unresolved) is appropriate.

On the other hand, for photoionized regions excited by sources with a power law photon distribution, a dominant part of the photoionizations is done by photons with $\nu \gg \nu_0$, and one then has $\sigma_H \ll \sigma_{\nu_0}$. This leads to much smaller values for the dimensionless parameter λ, resulting in transition regions with widths comparable to the Strömgren radius.

Because of this, the predicted spectra of photoionized regions in active galaxies can be dominated by the HII→HI transition region (see, e.g., Halpern & Steiner 1983 and Binette 1985). These spectra qualitatively resemble shock wave spectra (which are also produced in a transition region from ionized to neutral gas). For such low λ photoionization regions, our new solution gives the first available analytic description of the resulting ionization stratification.

The analytic approach described in this paper could be extended to more complex problems. One possibility would be to explore the ionization fraction in a constant pressure nebula (in which the density is position-dependent). A second possibility would be to consider the problem of a dusty HII region (see, e.g., Draine 2011), for which an analytic solution based on (Strömgren's 1939) $x \approx 1$ approximation (see § 1) has been derived by (Petrosian et al. 1972). This dusty region problem would be interesting in the context of dust in narrow line regions of active galactic nuclei (see, e.g., Binette et al. 1993).

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REFERENCES

Binette, L. 1985, A&A, 143, 334

Binette, L., Wang, J., Villar-Martín, M., Martin, P. G., Magris, C. G. 1993, ApJ, 414, 535

Draine, B. T. 2011, ApJ, 732, 100

Dyson, J. E., Williams, D. A. 1980, "The Physics of the Interstellar Medium", Manchester University Press, Manchester

Halpern, J. P., Steiner, J. E. 1983, ApJ, 269, L37

Petrosian, V., Silk, J., Field, G. B. 1977, ApJ, 177, L69

Strömgren, B. 1939, ApJ, 89, 526

Tremblin, P., Audit, E., Minier, V., Schmidt, W., Schneider, N. 2012, A&A, 546, 33

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