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ANALYTICAL SOLUTIONS FOR THE DYNAMICAL CLOCK A+ INDICATOR IN A TOY MODEL OF PURE DYNAMICAL FRICTION

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ABSTRACT

Blue straggler stars are more massive than the average star in globular clusters, as they originate from the merger of two stars. Consequently, they experience dynamical friction, progressively sinking to the cluster center. Recently, several indicators of the degree of dynamical relaxation of a globular cluster have been proposed, based on the observed radial distribution of blue straggler stars. The most successful is the Alessandrini indicator, or A+ for short, which is the integral of the cumulative distribution of the blue straggler stars minus that of a lighter reference population. A+ correlates with the dynamical age of a cluster both in realistic simulations and in observations. Here I calculate the temporal dependence of the A+ indicator analytically in a simplified model of the evolution of the blue straggler star distribution under dynamical friction only.

RESUMEN

Las estrellas rezagadas azules tienen masas mayores que las de las estrellas típicas de los cúmulos globulares, puesto que se originan de la fusión de dos estrellas. En consecuencia, experimentan fricción dinámica, y se acercan paulatinamente al centro del cúmulo. Se han propuesto recientemente varios indicadores del grado de relajamiento dinámico de un cúmulo globular basados en la distribución radial de las rezagadas azules. El más exitoso es el indicador de Alessandrini (A+), que es la integral de la distribución acumulada de las rezagadas azules menos la de estrellas menos masivas. A+ se correlaciona con la edad dinámica del cúmulo tanto en simulaciones realistas como en las observaciones. Aquí calculo analíticamente la dependencia con el tiempo del indicador A+, usando un modelo simplificado de la evolución de la distribución de las rezagadas azules, que toma en cuenta sólo la fricción dinámica.

Key Words: blue stragglers — globular clusters: general — methods: analytical

1. INTRODUCTION

Blue straggler stars are found in all globular clusters observed to date in the Milky way (Piotto et al. 2004). They are heavier than the average star in their host clusters, as they originate from stellar mergers either through direct collision (Hills & Day 1976) or by close-binary mass transfer (McCrea 1964; Knigge et al. 2009), or both (Davies et al. 2004; Mapelli et al. 2004). Since the first observations of a bimodality in the radial distribution of blue straggler stars when normalized to a reference population were done (Ferraro et al. 1993; Zaggia et al. 1997), attempts at understanding its origin and evolution have been made based on simulations run with different software and various levels of realism (Mapelli et al. 2004, 2006; Ferraro et al. 2012; Hypki & Giersz 2013; Miocchi et al. 2015; Hypki & Giersz 2017; Sollima & Ferraro 2019).

In a previous paper Pasquato et al. (2018) showed that the physical ingredients underlying the formation and motion of the minimum of the distribution are dynamical friction and diffusion respectively. While the two are connected as they ultimately arise from the same phenomenon, i.e. scatter with lighter background stellar particles, Pasquato et al. (2018) varied the diffusion coefficient and dynamical friction independently, showing that when diffusion is too

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strong a minimum does not reliably form, whereas if diffusion is too weak a clear-cut minimum forms but does not move outwards over time. This suggests that simulation schemes should be carefully assessed regarding their ability to correctly model the dynamical friction and diffusion phenomena in order to reproduce the observed evolution of the blue straggler star distribution minimum with increasing dynamical age.

In this context Alessandrini et al. (2016) introduced a new *dynamical clock* indicator which did not require a measurement of the position of the minimum of the normalized blue straggler star distribution, as it is based on the cumulative radial distribution of blue straggler stars compared to the cumulative distribution of some other class of reference stars. The Alessandrini et al. (2016) indicator (or A+ for short) was introduced in the context of direct N-body simulations, where it was shown that it increases with the dynamical age of simulated clusters, acting as a mass-segregation powered dynamical clock. Later, Lanzoni et al. (2016) measured a slightly modified version of the A+ indicator on a sample of 25 Galactic globular clusters, showing that it correlates with the cluster dynamical age measured in terms of a cluster's current relaxation time.

The A+ indicator is defined as the difference between the integral of the cumulative distribution of the blue straggler stars, expressed as a function of the logarithm of the cluster-centric radius, and that of a reference distribution. In the following I will obtain some of its properties analytically under simplifying assumptions.

2. CALCULATIONS

2.1. *A Toy Model of Dynamical Friction*

I model blue straggler stars as a population of equal mass particles in circular orbits in a spherically symmetric fixed gravitational potential. The radius r of each orbit evolves due to dynamical friction, as

$$\dot{r} = -\frac{r}{\tau(r)} = -v(r), \tag{1}$$

where r is the distance from the center and $\tau(r)$ is a positive, monotonically increasing function of r, representing the dynamical friction timescale at radius r.

Equation 1 shows that orbital radii contract with an instantaneous velocity $v(r) > 0$ that depends only on r. It can be integrated, obtaining

$$\int_{r_0}^{r} \frac{\tau(x)dx}{x} = -t, \tag{2}$$

where r_0 is the initial value of the radius at time $t = 0$ and r is its current value at time t. In general $r_0 > r$ because the radii contract over time. If the function $\tau(x)$ is known, the integral can be calculated and r can be obtained as a function of r_0 and t:

$$r(r_0, t) = I^{-1}(I(r_0) - t), \tag{3}$$

where the primitive

$$I(r) = \int \frac{\tau(x)dx}{x} \tag{4}$$

is a monotonically increasing function because $\tau(x)/x$ always is positive. Consequently, it is invertible. Note also that $r(r_0, t)$ is a monotonically decreasing function of t for every $t > 0$ and for every r_0, i.e. the orbit radii keep shrinking over time. This can be shown by writing

$$I(r_0) - t < I(r_0), \tag{5}$$

which holds for every $t > 0$, and applying I^{-1}, which is also monotonic, to both sides, yields

$$r(r_0, t) = I^{-1}(I(r_0) - t) < I^{-1}(I(r_0)) = r_0. \tag{6}$$

Similarly to equation 3,

$$r_0(r, t) = I^{-1}(I(r) + t), \tag{7}$$

also holds.

I now denote with $N(r, t)$ the cumulative distribution of particles at a given time as a function of radius. This is by construction such that $N(0, t) = 0$ and $\lim_{r\to\infty} N(r, t) = 1$ for all t. If for any two particles at time $t = 0$ the condition $r_{01} < r_{02}$ held, then at any subsequent t, $r_1(t) < r_2(t)$ would also hold: particles never cross. This holds due to the uniqueness of the solution of first-order ordinary differential equations. Therefore, all particles within a distance r of the center at time t were within a distance $r_0(r, t)$ at time 0, as can be seen by placing an imaginary particle exactly at r and observing that no other particle ever crosses its path. In other words, Lagrangian radii behave exactly like particle radii. So

$$N(r, t) = N(r_0(r, t), 0), \tag{8}$$

the number of particles that had a radius less than a given r_0 at the beginning still have a radius less than $r(r_0, t)$ at time t. This can be rewritten as

$$N(r, t) = N(I^{-1}(I(r) + t), 0), \tag{9}$$

which, given knowledge of the function I is a general solution for $N(r, t)$. Thus $\tau(r)$ fully determines $N(r, t)$ given an initial $N(r, 0)$.

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2.2. *Recovering the A+ Indicator*

In the following I will assume that the reference population of stars to which the blue stragglers are compared to build the A+ indicator initially shares the same distribution as the blue stragglers and does not evolve.

Under this assumption it is trivial to obtain the evolution of the (three-dimensional) A+ indicator from equation 9. I will write $s = \log r$, so that

$$N(r,t) = N(I^{-1}(I(e^s) + t), 0), \tag{10}$$

and the A+ indicator becomes

$$A^{+}(t) = \int_{-\infty}^{+\infty} N(I^{-1}(I(e^s)+t),0)ds - \int_{-\infty}^{+\infty} N(e^s,0)ds. \tag{11}$$

2.3. *Monotonicity*

Note that at time $t_2 > t_1$

$$A^{+}(t_2) - A^{+}(t_1) = \int_{-\infty}^{+\infty} \left[N(I^{-1}(I(e^s) + t_2), 0) - N(I^{-1}(I(e^s) + t_1), 0)\right] ds, \tag{12}$$

and the integrand

$$N(I^{-1}(I(e^s) + t_2), 0) - N(I^{-1}(I(e^s) + t_1), 0) \tag{13}$$

is positive for every s, because $I^{-1}(I(e^s) + t_2) > I^{-1}(I(e^s) + t_1)$ as the two terms represent, per equation 7, the initial radius of a particle that is at $r = e^s$ at t_2 and t_1 respectively: a particle that took more time ($t_2 > t_1$) to fall to r was further away at the beginning. This implies that $A^{+}(t)$ is a monotonically increasing function of time, i.e. a working dynamical clock.

2.4. *A+ Linear Dependence in Globular Cluster Cores*

While equation 7 can be solved numerically for any $\tau(r)$, some choices of $\tau(r)$ will lead to a simple analytical solution. For example, following equation 1 of Mapelli et al. (2004) I take

$$\tau(r) = \tau(0)\frac{\sigma^3(r)\rho(0)}{\sigma^3(0)\rho(r)}, \tag{14}$$

where σ is the velocity dispersion of background stars at radius r and ρ is their number density. For a Plummer model this works out as

$$\tau(r) = \tau(0)\left(1 + \frac{r^2}{a^2}\right)^{7/4}, \tag{15}$$

where a is the model scale radius and $\tau(0)$ the scale time for dynamical friction at the center. Equation 4 is solved exactly, for this dependence, by

$$I(u) = \tau(0)\left[\frac{1}{2}\log\left(\frac{u-1}{u+1}\right) + \arctan(u) + \frac{2}{7}u^7 + \frac{2}{3}u^3\right], \tag{16}$$

where

$$u = \left(1 + \frac{r^2}{a^2}\right)^{1/4} > 1, \tag{17}$$

which unfortunately cannot be inverted in terms of simple functions. However for small radii equation 15 reduces to a constant, so equation 4 becomes trivially

$$I(r) = \tau(0)\log(r/a), \tag{18}$$

and, with reference to equation 7

$$r = r_0 e^{-t/\tau(0)}, \tag{19}$$

so

$$N(r,t) = N(ae^{\log(r/a)+t/\tau(0)}, 0) = N(re^{t/\tau(0)}, 0). \tag{20}$$

As the central regions of a Plummer model have approximately constant density $\rho(0)$, I can take at time $t = 0$

$$N_c(r,0) = 4\pi\rho(0)r^3, \tag{21}$$

with a radial cutoff at

$$r_{c0} = (4\pi\rho(0))^{-1/3}, \tag{22}$$

after which $N_c(r,0)$ becomes identically 1. At time t the radius at which $N_c(r,t)$ becomes identically 1 is

$$r_c = r_{c0}e^{-t/\tau(0)}. \tag{23}$$

Therefore

$$A^{+}(t) = \int_{-\infty}^{\log r_c} N_c(e^{s+t/\tau(0)}, 0)ds + \int_{\log r_c}^{\log r_{c0}} 1 ds - \int_{-\infty}^{\log r_{c0}} N_c(e^s, 0)ds, \tag{24}$$

which simplifies to

$$A^{+}(t) = \log\frac{r_{c0}}{r_c} = \frac{t}{\tau(0)}. \tag{25}$$

This result actually generalizes to any non-constant initial density as long as equation 19 holds, because of the interplay between the logarithm in the definition of the A+ indicator and the exponential dependence of equation 19, which leads to the first and the third term in equation 24 canceling out. Thus the A+ indicator should evolve linearly with time if the dynamical friction timescale is constant with radius.

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3. CONCLUSIONS

Working within a pure dynamical friction picture, under a set of simplifying assumptions, I have shown that the Alessandrini et al. (2016) A+ indicator evolves monotonically in time and I have found an analytical solution for its time dependence. I worked out the case of a dynamical friction timescale that is constant with radius, which results in the A+ indicator increasing linearly with time. Monotonicity is an interesting result, as it proves that the A+ indicator is effectively a *dynamical clock*, as previously claimed by Alessandrini et al. (2016) based on the results of a set of direct N-body simulations. As my simple model neglects diffusion, which was instead treated numerically by Pasquato et al. (2018), I showed that the A+ indicator still works as a dynamical clock even in the absence of diffusion.

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