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INVESTIGATING THE HYPERBOLIC AND HYBRID SCALAR FIELD COSMOLOGIES WITH VARYING COSMOLOGICAL CONSTANT IN $F(R,T)$ GRAVITY

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ABSTRACT

This paper investigated two scalar field cosmological models in $f(R,T)$ gravity with cosmic transit and varying cosmological constant $\Lambda(t)$. The cosmological constant tends to have a tiny positive value at the current epoch. The scalar field pressure p_ϕ shows a sign flipping for normal scalar field. For the phantom field, the scalar potential $V(\phi)$ is negative and the energy density $\rho_\phi = E_k + V$ takes negative values when the equation of state parameter ω_ϕ is less than -1. The WEC, $\rho = \sum_i \rho_i \geq 0$ and $p_i + \rho_i \geq 0$, is not violated but with an instability for the second model at late times. For a scalar field ϕ, the condition $\rho_\phi + p_\phi = \rho_\phi(1+\omega_\phi) = 2E_k \geq 0$ allows for $\rho_\phi < 0$ if $\omega_\phi < -1$. The causality and energy conditions have been discussed for both models. The cosmology in both models was studied using a given function $a(t)$ derived from the desired cosmic behavior, which is the opposite of the traditional view.

RESUMEN

Investigamos dos modelos cosmológicos de campo escalar suponiendo gravedad $f(R,T)$, con tránsito cósmico y constante cosmológica $\Lambda(t)$ variable. La constante cosmológica tiende a un valor pequeño y positivo en el presente. La presión del campo escalar p_ϕ cambia de signo para un campo escalar normal. Para el campo fantasma, el potencial escalar $V(\phi)$ es negativo y la densidad de energía $\rho_\phi = E_k + V$ adquiere valores negativos cuando el parámetro ω_ϕ de la ecuación de estado es menor que -1. No se viola la WEC, $\rho = \sum_i \rho_i \geq 0$ y $p_i + \rho_i \geq 0$, pero se obtiene una inestabilidad en el segundo modelo, a tiempos tardíos. Para un campo escalar ϕ, la condición $\rho_\phi + p_\phi = \rho_\phi(1+\omega_\phi) = 2E_k \geq 0$ permite una $\rho_\phi < 0$ si $\omega_\phi < -1$. Se discuten las condiciones de causalidad y energía para ambos modelos. Se estudia la cosmología en ambos modelos con una función dada $a(t)$ derivada del comportamiento cósmico deseado, lo cual es contrario a la visión tradicional.

Key Words: cosmological parameters — cosmology: theory — dark energy

1. INTRODUCTION

Accelerated cosmic expansion (Percival et al. 2001; Stern et al. 2010) has become a basic motivation for a variety of modified gravitational theories (Nojiri & Odintsov 2006; Nojiri et al. 2008; Ferraro & Fiorini 2007; Bengochea & Ferraro 2009; De Felice & Tsujikawa 2010; Alves et al. 2011; Maeder 2017; Gagnon & Lesgourgues 2011; Ahmed 2009, 2010; Ahmed & Pradhan 2022; Ahmed & Kamel 2021). In order to find a satisfactory explanation, an exotic form of energy with negative pressure, called dark energy, was hypothesized. Several dynamical scalar fields models of dark energy were introduced such as Quintessence, Phantom and Tachyons (Tsujikawa 2013; Kamenshchik et al. 2001; Caldwell 2002; Chiba et al. 2000; Sen 2002; Arkani-Hamed et al. 2004; Ahmed et al. 2023). For a zero curvature FRW universe driven by a scalar field ϕ, Einstein's equations are

$$3H^2 = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad \dot{H} = -\frac{1}{2}\dot{\phi}^2, \quad \ddot{\phi} + 3H\dot{\phi} + V' = 0. \tag{1}$$

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With units $8\pi M_{Pl}^{-2} = c = 1$. $H = \frac{\dot{a}}{a}$ is the Hubble parameter and $V(\phi)$ is the potential. The prime denotes differentiation with respect to ϕ, and the dots denote differentiation with respect to t. While this nonlinear system is insoluble in general, progress can be made through postulating a particular form of the scale factor $a(t)$ and then obtaining the form of both $\phi(t)$ and $V(\phi)$ (Barrow & Parsons 1995; Ellis & Madsen 1991). In Banerjee & Pavón (2001), it was shown that a minimally coupled scalar field in Brans-Dicke theory leads to an accelerating universe. A power function forms of the scale factor a and the scalar field ϕ were assumed as

$$a(t) = a_1 t^{\alpha}, \qquad \phi(t) = \phi_1 t^{\beta}, \tag{2}$$

with a_1, ϕ_1, α and β constants. An accelerated expansion was also achieved in a modified Brans-Dicke theory through considering the following power-law form of both a and ϕ (Bertolami & Martins 2000).

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{\alpha}, \qquad \phi(t) = \phi_0 \left(\frac{t}{t_0}\right)^{\beta}. \tag{3}$$

Cosmology in the scalar-tensor $f(R,T)$ gravity has been studied in Gonçalves et al. (2022) where three particular forms of $a(t)$ have been used.

1.1. *Negative Potentials and Energy Densities*

The case of negative potential cosmologies has become interesting after the prediction of Ads spaces in string theory and particle physics. Negative potentials also exist in ekpyrotic and cyclic cosmological models in which the universe goes from a contracting to an accelerating phase (Steinhardt & Turok 2002; Khoury et al. 2001). They are commonly predicted in particle physics, supergravity and string theory where the general vacuum of supergravity has a negative potential. It has also been suggested that negative potentials lead to an explanation of the cosmological scale in terms of a high energy scale such as the supersymmetry breaking scale or the electroweak scale (Garriga & Vilenkin 2000). A detailed discussion of scalar field cosmology with negative potentials was carried out in Felder et al. (2002). The effect of negative energy densities on classical FRW cosmology has been investigated in Nemiroff et al. (2015) where the total energy density can be expanded as

$$\rho = \sum_{n=-\infty}^{\infty} \rho_n^{+} a^{-n} + \sum_{m=-\infty}^{\infty} \rho_m^{-} a^{-m}, \tag{4}$$

where ρ_n^{+} is the familiar positive energy density and ρ_m^{-} is the negative cosmological energy density. The cosmic evolution with negative energy densities was also examined in Saharian et al. (2022) where vacuum polarization was mentioned as an example for a gravitational source with $\rho < 0$ that may have played a significant role in early cosmic expansion.

An interesting study was carried out in De La Macorra & Germán (2022) where the equation of state parameter is negative ($\omega_{\phi} = p_{\phi}/\rho_{\phi} < -1$) with no violation of the weak energy condition ($\rho = \sum_i \rho_i \geq 0$ & $p_i + \rho_i \geq 0$) which requires a negative potential $V(\phi) < 0$. It has been shown that $\rho_{\phi} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ becomes negative with $\omega_{\phi} < -1$, the negative ρ_{ϕ} leads to a small value of the cosmological constant. However, while cosmic expansion exists in such scenario, the negative potential V leads to a collapsing universe.

The classical energy conditions are "the null energy condition (NEC) $\rho + p \geq 0$; weak energy condition (WEC) $\rho \geq 0$, $\rho + p \geq 0$; strong energy condition (SEC) $\rho + 3p \geq 0$ and dominant energy condition (DEC) $\rho \geq |p|$". Since the SEC implies that gravity should always be attractive, this condition fails in the accelerating and inflation epochs (Visser 1997a,b). As was mentioned in Barceló et al. (2002), even the simplest scalar field theory we can write down violates the SEC. The NEC is the most fundamental energy condition on which the singularity theorems, and other key results, are based (Alexandre & Polonyi 2021). If the NEC is violated, all other point-wise energy conditions (ECs) are automatically violated. A very useful discussion about the validity of classical linear ECs was given in Barceló et al. (2002) where it was shown that these classical conditions cannot be valid in general situations. The scalar field potential $V(\phi)$ is restricted by the ECs where the scalar field ϕ (with $\rho_{\phi} = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ & $p_{\phi} = \frac{1}{2}\dot{\phi}^2 - V(\phi)$) satisfies the NEC for any $V(\phi)$, the WEC if and only if $V(\phi) \geq -\frac{1}{2}\dot{\phi}^2$, the DEC if and only if $V(\phi) \geq 0$, the SEC if and only if $V(\phi) \leq \dot{\phi}^2$. The detailed proof of this theorem can be found in Westmoreland (2013).

1.2. $\Lambda(t)$ *Models*

A new model for the time-dependent cosmological constant $\Lambda(t)$ was proposed in Lopez & Nanopoulos (1996) using the following ansatz

$$\Lambda = \frac{\Lambda_{Pl}}{(t/t_{Pl})^2} \propto \frac{1}{t^2}, \tag{5}$$

Λ starts at the Planck time as $\Lambda_{Pl} \simeq M_{Pl}^2$ and leads to the value $\Lambda_0 \approx 10^{-120} M_{Pl}^2$ for the current epoch. The decay of $\Lambda(t)$ during inflation and as Bose condensate evaporation was studied in Dymnikova &

Khlopov (2001, 2000). Other models for $\Lambda(t)$ have been suggested in Basilakos et al. (2009; Pan (2018); Oikonomou et al. (2017); Ahmed & Alamri (2018, 2019a). The following ansatz was first introduced in Basilakos et al. (2009) where a variety of cosmologically relevant observations were used to put strict constraints on $\Lambda(t)$ models

$$\Lambda(H) = \lambda + \alpha H + 3\beta H^2, \tag{6}$$

where H is the Hubble parameter, λ, α and β are constants. It has been found in Pan (2018); Basilakos et al. (2013); Gómez-Valent & Solà (2015) that the zero value of λ does not agree with observations, while $\lambda \neq 0$ behaves like the ΛCDM model at late-time. Examples of varying Λ models in terms of the Hubble parameter H are (Pan 2018)

$$\Lambda(H) = \beta H + 3H^2 + \delta H^n, \quad n \in R - \{0, 1\}, \tag{7}$$
$$\Lambda(H, \dot{H}, \ddot{H}) = \alpha + \beta H + \delta H^2 + \mu \dot{H} + \nu \ddot{H}. \tag{8}$$

A generalized holographic dark energy model where the effective cosmological constant depends on H and its derivatives was proposed in Nojiri et al. (2021, 2020, 2022a).

1.3. $f(R,T)$ *Modified Gravity*

The action of $f(R,T)$ modified gravity is given as (Harko et al. 2011)

$$S = \int \left(\frac{f(R,T)}{16\pi G} + L_m \right) \sqrt{-g}\, d^4x, \tag{9}$$

where L_m is the matter Lagrangian density. $f(R,T)$ is an arbitrary function of the Ricci scalar R and the trace T of the energy-momentum tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = g_{\mu\nu} L_m - 2\frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{10}$$

Varying the action (9) gives

$$\begin{aligned} f_R(R,T)R_{\mu\nu} - \frac{1}{2} f(R,T) g_{\mu\nu} \\ +(g_{\mu\nu}\diamond - \nabla_\mu \nabla_\nu) f_R(R,T) \\ = 8\pi T_{\mu\nu} - f_T(R,T)T_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}, \end{aligned} \tag{11}$$

where $\diamond = \nabla^i \nabla_i$, $f_R(R,T) = \frac{\partial f(R,T)}{\partial R}$, $f_T(R,T) = \frac{\partial f(R,T)}{\partial T}$ and ∇_i denotes the covariant derivative. $\Theta_{\mu\nu}$ is given by

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu} L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}. \tag{12}$$

The cosmological equations for $f(R,T) = R + 2h(T)$ with cosmological constant Λ considering a scalar field ϕ coupled to gravity were given in Aygün et al. (2018) as

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = 4\pi\epsilon\dot{\phi}^2 - 8\pi V(\phi) + \mu\epsilon\dot{\phi}^2 - 4\mu V(\phi) - \Lambda, \tag{13}$$
$$\frac{3\dot{a}^2}{a^2} = -4\pi\epsilon\dot{\phi}^2 - 8\pi V(\phi) - \mu\epsilon\dot{\phi}^2 - 4\mu V(\phi) - \Lambda, \tag{14}$$

where $h(T) = \mu T$ and μ is a constant; $\epsilon = \pm 1$ corresponding to normal and phantom scalar fields respectively. In the current work, two cosmological models in modified $f(R,T)$ gravity were investigated using a given scale factor $a(t)$ deduced from the desired cosmic behavior which is the opposite of the conventional viewpoint. Such ad hoc approach to the cosmic scale factor and cosmological scalar fields was widely used by many authors in various theories (Ellis & Madsen 1991; Chervon et al. 1997; Sen & Sethi 2002; Maharaj et al. 2017; Silva & Santos 2013; Ahmed & Alamri 2019b; Sazhin & Sazhina 2016; Ahmed et al. 2020; Ahmed 2020; Ahmed & Kamel 2021; Ahmed & Pradhan 2020; Nojiri et al. 2022b). We will make use of the following hyperbolic and hybrid scale factors:

$$a(t) = A \sinh^{\frac{1}{n}}(\eta t) \qquad , \qquad a(t) = a_1 t^{\alpha_1} e^{\beta_1 t}, \tag{15}$$

where A, η, n, $a_1 > 0$, $\alpha_1 \geq 0$ and $\beta_1 \geq 0$ are constants. The first scale factor generates a class of accelerating models for $n > 1$; the models also exhibit a phase transition from the early decelerating epoch to the present accelerating era in good agreement with recent observations. The second hybrid ansatz is a mixture of power-law and exponential-law cosmologies, and can be regarded as a generalization of each of them. The power-law cosmology can be obtained for $\beta_1 = 0$, and the exponential-law cosmology can be obtained for $\alpha_1 = 0$. New cosmologies can be explored for $\alpha_1 > 0$ and $\beta_1 > 0$. A generalized form of the hybrid scale factor has been proposed in Nojiri et al. (2022b); Odintsov et al. (2021) to unify the cosmic evolution of the universe from a non-singular bounce to the viable dark energy

$$a(t) = \left[1 + a_0 \left(\frac{t}{t_0}\right)^2\right]^{\frac{1}{3(1+\omega)}} \exp\left[\frac{1}{(\alpha - 1)} \left(\frac{t_s - t}{t_0}\right)^{1-\alpha}\right], \tag{16}$$

where ω, α and t_s are various parameters. Setting $t_0 = 1$ billion years, this can be re-written as the product of two scale factors

$$a(t) = \left[1 + a_0 t^2\right]^{\frac{1}{3(1+\omega)}} \times \exp\left[\frac{1}{(\alpha - 1)} (t_s - t)^{1-\alpha}\right]. \tag{17}$$

In the current work, we are going to use the ansatz (6) for the time varying cosmological constant, which leads to a very tiny positive value of Λ at the current epoch as suggested by observations (Perlmutter et al. 1999; Tonry et al. 2003).

2. MODEL 1

Starting with the hyperbolic solution in (15), which gives the desired behavior of the deceleration and jerk parameters, we obtain the Hubble, deceleration, and jerk parameters as:

$$H = \frac{\eta}{n}\coth(\eta t), \quad q = -\frac{\ddot{a}a}{\dot{a}^2} = \frac{-\cosh^2(\eta t) + n}{\cosh^2(\eta t)},$$
$$j = \frac{\dddot{a}}{aH^3} = 1 + \frac{2n^2 - 3n}{\cosh^2(\eta t)}. \quad (18)$$

In order to solve the system of equations (13) and (14) for the scalar field and the potential, we utilize the hyperbolic scale factor in (15) along with the time-dependent anstaz for the cosmological constant (6). Then, we will have a system of two equations in two unknowns which we have solved using Maple software and have obtained

$$\phi(t) = \frac{\mp\ln(e^{\eta t}+1) \pm \ln(e^{\eta t}-1)}{\sqrt{-2\epsilon(4\pi+\mu)}} + \phi_0, \quad (19)$$

$$V(t) = -\frac{\left(\eta^2(1+3\beta)\coth^2(\eta t) + 2\eta\alpha\coth(\eta t) + 2(\eta^2+4\lambda)\right)}{16(2\pi+\mu)}, \quad (20)$$

$$V(\phi) = -\frac{\left((3\beta+1)\,\eta^2\chi^2 + 4\eta\alpha\chi + 2\eta^2(3\beta+5)\right)}{64(2\pi+\mu)} - \frac{\left(16\lambda + 4\eta\alpha\chi^{-1} + \eta^2\chi^{-2}(3\beta+1)\right)}{64(2\pi+\mu)}, \quad (21)$$

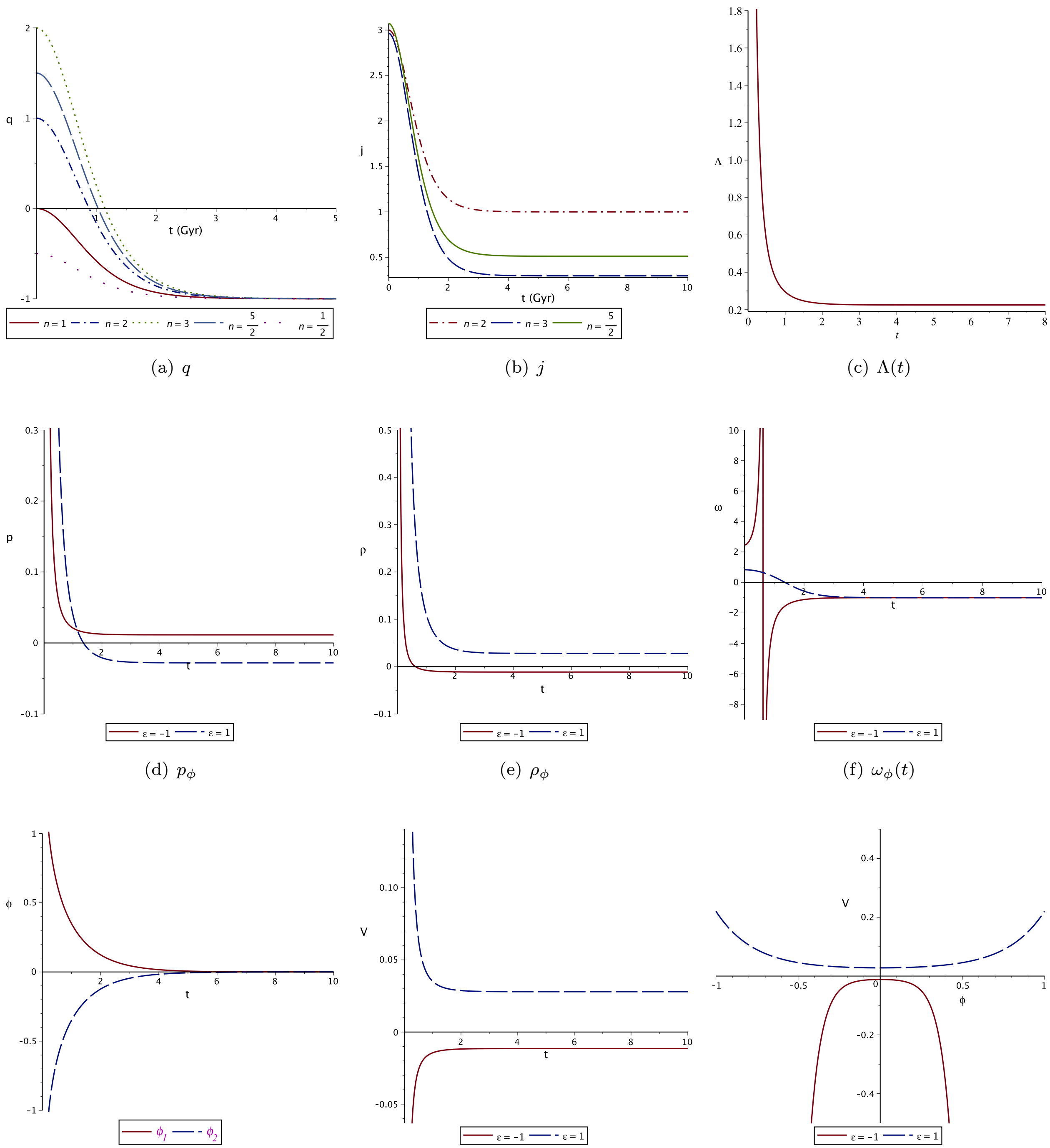
where $\chi \equiv e^{(\phi_0-\phi)\sqrt{-2\epsilon(4\pi+\mu)}}$ and we have used $t(\phi) = \frac{1}{\eta}\ln(\mp\frac{1+\chi}{\chi-1})$ to get the expression for $V(\phi)$. The expression for $\phi(t)$ shows that ϵ can be -1 provided that $(4\pi+\mu) > 0$, and it can be $+1$ provided that $(4\pi+\mu) < 0$. Plotting $t(\phi)$ leads to same graph for both signs (Sen 2002). We also obtain the same expressions for $V(\phi)$ (Ahmed et al. 2023), energy density ρ and pressure p for both ϕ solutions. Actually, Figure 1(g) shows that both solutions for ϕ, although they have a different start, unite in one solution. We can use $\phi_0 = 0$ without loss of generality. Recalling that $\rho_\phi = E_k + V$ and $p_\phi = E_k - V$ we obtain

$$p_\phi(t) = -\frac{\eta^2 e^{2\eta t}}{\epsilon(4\pi+\mu)(e^{\eta t}+1)^2(e^{\eta t}-1)^2} - V(t),$$
$$\rho_\phi(t) = -\frac{\eta^2 e^{2\eta t}}{\epsilon(4\pi+\mu)(e^{\eta t}+1)^2(e^{\eta t}-1)^2} + V(t). \quad (22)$$

The evolution of the cosmological constant in this work agrees with observations where it has a very tiny positive value at the current epoch (Figure 1c). The expressions for the parameters q, j and the cosmological constant in equation (6) are all independent of ϵ. The rest of the parameters are all plotted for $\epsilon = \pm 1$. For $\epsilon = +1$, which corresponds to a normal scalar field, the scalar field pressure p_ϕ changes sign from positive to negative. We can also see that $V(\phi)$, $V(t)$ and ρ_ϕ are all positive where both $V(t)$ and ρ_ϕ tend to ∞ as $t \to 0$. For $\epsilon = -1$, which corresponds to a phantom scalar field, the pressure $p_\phi > 0$ all the time while ρ_ϕ takes negative values when $\omega_\phi < -1$ with a negative scalar potential V. In the literature, it is known that the vacuum phantom energy has some unusual physical properties such as the increasing vacuum energy density, violation of the DEC $\rho+p < 0$ and the superluminal sound speed (González-Díaz 2004).

According to the WEC, the total energy density and pressure should follow the inequalities $\rho + p = \rho(1+\omega) \geq 0$ and $\rho \geq 0$. For a scalar field ϕ, the condition $\rho_\phi + p_\phi = \rho_\phi(1+\omega_\phi) = 2E_k \geq 0$ allows for $\rho_\phi < 0$ if $\omega_\phi < -1$ as long as the total energy density $\rho \geq 0$ with the total equation of state parameter $\omega > -1$. In general, the phantom energy does not obey the WEC where it has $\rho_{ph} > 0$ but $\rho_{ph} + p_{ph} = \rho_{ph}(1+\omega_{ph}) = 2E_k < 0$ which means that the phantom field has a negative (non-canonical) kinetic term (De La Macorra & Germán 2004). Testing the classical energy conditions (Visser 1997b) shows that both the null and the dominant are satisfied all the time. The highly restrictive SEC $\rho + 3p \geq 0$ is violated as expected where we have a source of repulsive gravity represented by the negative pressure, which can accelerate cosmic expansion. Because the strong condition implies that gravity should always be attractive, it is expected to be violated during any accelerating epoch dominated by a repulsive gravity effect such as cosmic inflation. In addition to the ECs, the sound speed causality condition $0 \leq \frac{dp}{d\rho} \leq 1$ is satisfied only for $\epsilon = +1$.

The possible values of the parameters in the figures are restricted by observations, whereas the theoretical model should predict the same behavior obtained by observations. For that reason, we have to

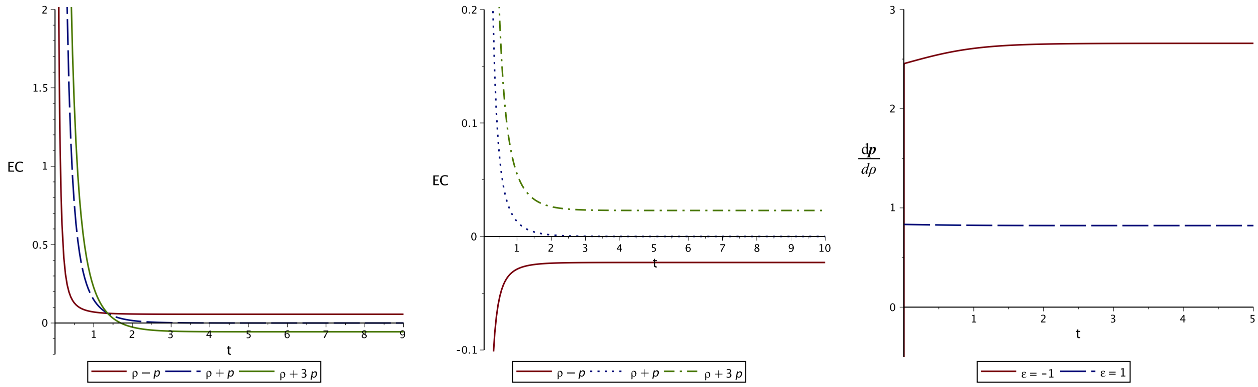


(a) q (b) j (c) $\Lambda(t)$

(d) p_ϕ (e) ρ_ϕ (f) $\omega_\phi(t)$

(g) $\phi(t)$ (h) $V(t)$ (i) $V(\phi)$

Fig. 1. The hyperbolic solution: (a) The deceleration parameter q shows a decelerating-accelerating cosmic transit. (b) The jerk parameter approaches unity at late-times where the model tends to a flat ΛCDM model. (c) The cosmological constant reaches a very tiny positive value at the current epoch. (d), (e) & (f) show p_ϕ, ρ_ϕ and ω_ϕ for $\epsilon = \pm 1$. For the phantom case, the energy density $\rho_\phi = E_k + V < 0$ when $\omega_\phi < -1$. (g) The two solutions of $\phi(t)$ obtained in Sen (2002). (h) The scalar potential evolution with time. (g) scalar potential V verses ϕ . Here $n = 2, \eta = 1, \phi_0 = 0, A = \lambda = \beta = \alpha = 0.1$, $\mu = 15$ for $\epsilon = -1$ and -15 for $\epsilon = 1$. The color figure can be viewed online.



(a) $\epsilon = +1$ (b) $\epsilon = -1$ (c) dp/dt

Fig. 2. ECs and sound speed for the hyperbolic model. Superluminal sound speed for the phantom field. The color figure can be viewed online.

fine-tune the parameters' values to agree with observational results. We have taken $n = 2$ as it allows for a decelerating-accelerating cosmic transit and also allows the jerk parameter j to approach unity at late-times in agreement with the standard ΛCDM model. The constants A, η, and the integration constant ϕ_0 are arbitrary and we have chosen the values 0.1, 1 and 0 respectively without loss of generality. The value of the constant μ has been adjusted such that the quantity under the quadratic root in (19) is always positive for both normal and phantom fields. If we choose $\mu = 15$, then $(4\pi + \mu) > 0$ for the normal field where $\epsilon = +1$. For the phantom field with $\epsilon = -1$, we choose $\mu = -15$ so $(4\pi + \mu) < 0$ and then $-2\epsilon(4\pi + \mu) > 0$. As we have indicated in § 1.2, the zero value of λ does not agree with observations while $\lambda \neq 0$ behaves like the ΛCDM model at late-time. Based on this, we have chosen the non-zero value 0.1 for λ, β and α.

3. MODEL 2

Considering the second hybrid scale factor in (15), which also leads to the desired behavior of both q and j (Ahmed 2020), we get the expressions for H, q and j as:

$$H = \beta_1 + \frac{\alpha_1}{t}, q = \frac{\alpha_1}{(\beta_1 t + \alpha_1)^2} - 1,$$
$$j = \frac{{\alpha_1}^3 + (3\beta t - 3)\,{\alpha_1}^2 + \left(3\beta^2 t^2 - 3\beta t + 2\right)\alpha_1 + \beta^3 t^3}{\left(\beta\, t + \alpha_1\right)^3}. \tag{23}$$

For the scalar field and the potential, making use of (6), we get

$$\phi(t) = \pm \frac{\sqrt{-\epsilon(4\pi + \mu)\alpha_1}\ln t}{\epsilon(4\pi + \mu)} + C_1, \tag{24}$$

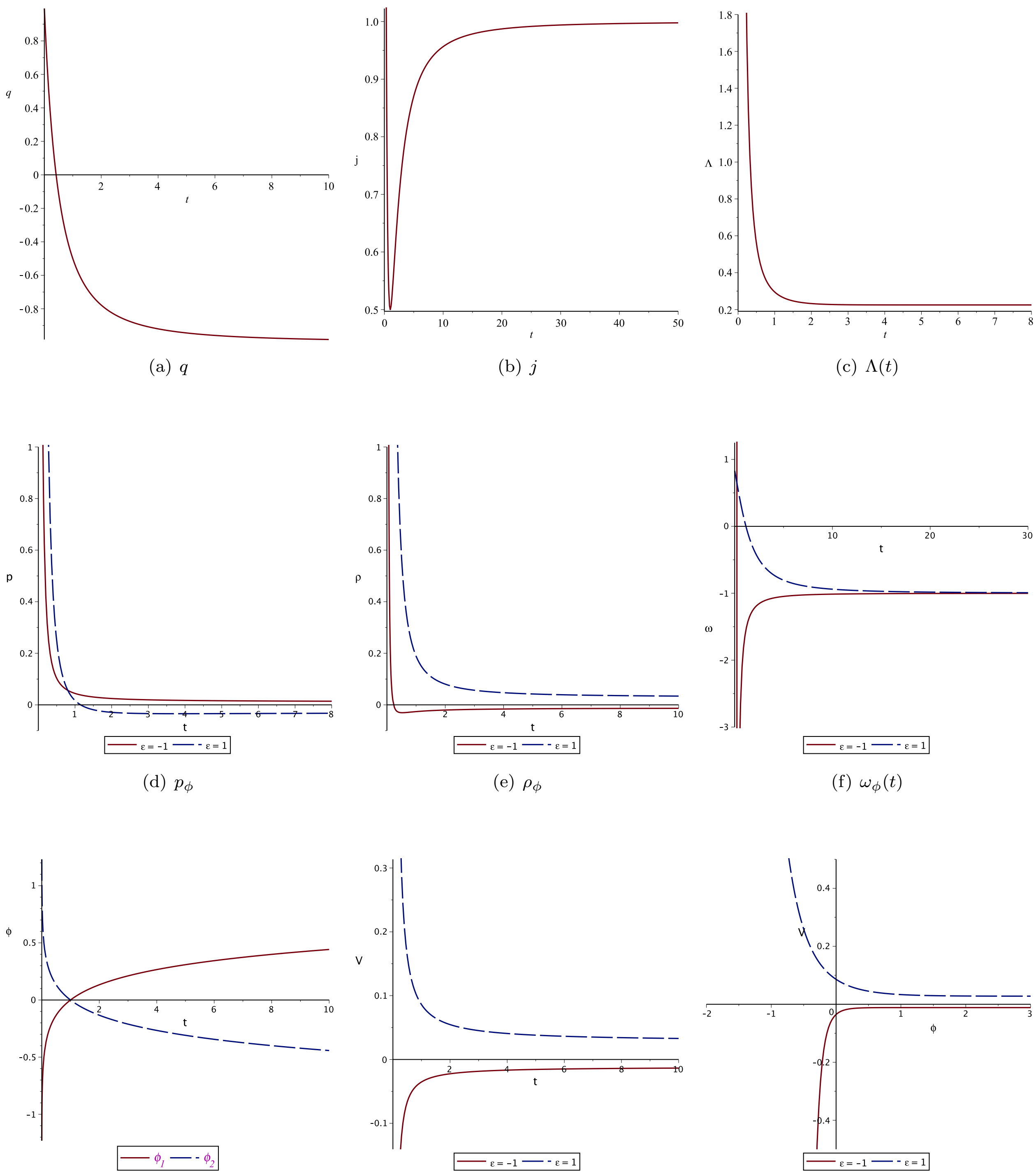
$$\begin{aligned} V(t) \quad &= \frac{\left(3\beta_1^2(\beta_0 + 1) + \alpha_0\beta_1 + \lambda_0\right)t^2}{-4(\mu + 2\pi)t^2} + \\ &\frac{\left(6\alpha_1\beta_1\left(\beta_0 + 1\right) + \alpha_0\alpha_1\right)t}{-4(\mu + 2\pi)t^2} + \\ &\frac{3\alpha_1^2\left(\beta_0 + 1\right) + \alpha_1}{-4(\mu + 2\pi)t^2}, \end{aligned} \tag{25}$$

$$\begin{aligned} V(\phi) \quad &= \frac{3\left(\beta_1^2 + \alpha_1^2\right)\left(\beta_0 + 1\right) + \alpha_0\beta_1}{-4(\mu + 2\pi)} + \\ &\frac{\xi^{-1}\alpha_1\left(6\beta_0\beta_1 + \alpha_0 + 6\beta_1\right) + \lambda_0 - \alpha_1}{-4(\mu + 2\pi)}, \end{aligned} \tag{26}$$

where $\xi = e^{\frac{\epsilon(|C_1 - \phi)(4\mu + \pi)}{\sqrt{-\epsilon\alpha_1(4\mu + \pi)}}} = t(\phi)$. Plotting $t(\phi)$ leads to same graph for both signs. Also, both solutions for ϕ give the same expressions for ρ and p as

$$p(t) = \frac{-\alpha_1}{2\epsilon(4\pi + \mu)t^2} - V(t), \quad \rho(t) = \frac{-\alpha_1}{2\epsilon(4\pi + \mu)t^2} + V(t). \tag{27}$$

In comparison to the first hyperbolic model, a similar behavior has been obtained for different parameters in the hybrid model. For $\epsilon = +1$, p_ϕ changes sign from positive to negative indicating a cosmic transit. $V(\phi)$, $V(t)$ and ρ_ϕ are > 0 where both $V(t)$ and $\rho_\phi \to \infty$ as $t \to 0$. For $\epsilon = -1$, p_ϕ is always positive while ρ_ϕ takes negative values when $\omega_\phi < -1$ with a negative scalar potential V. In the current work, we argue that the WEC is not violated for the two models considered with an instability at late-times for the second model, which now can be seen in Figure 4(c). The WEC, asserting that the total energy density ρ must be non-negative, is challenged by the notion that a negative term in the energy density can coexist if the overall energy density remains

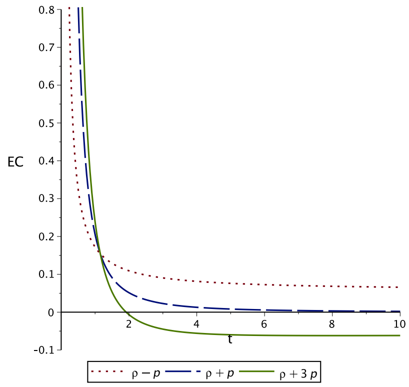


(a) q (b) j (c) $\Lambda(t)$

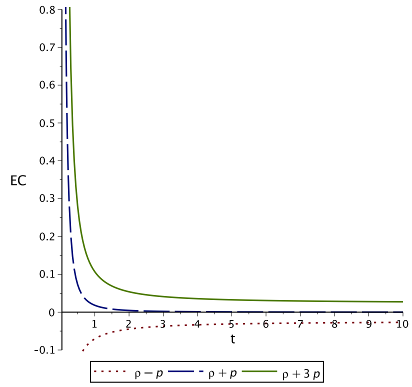
(d) p_ϕ (e) ρ_ϕ (f) $\omega_\phi(t)$

(g) $\phi(t)$ (h) $V(t)$ (i) $V(\phi)$

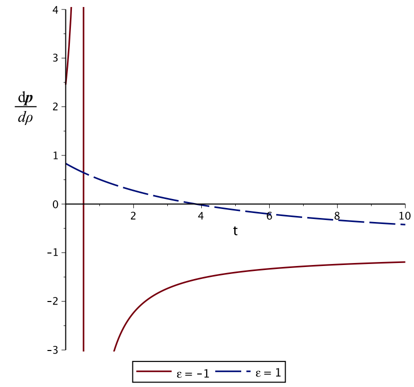
Fig. 3. The second model: (a) A decelerating-accelerating cosmic transit. (b) The jerk parameter $j = 1$ at late-times. (c) The cosmological constant reaches a very tiny positive value at the current epoch. (d), (e), & (f) show p_ϕ, ρ_ϕ and ω_ϕ for $\epsilon = \pm 1$. For the phantom case, the energy density $\rho_\phi < 0$ when $\omega_\phi < -1$. (g) The two solutions of $\phi(t)$ obtained in Sen (2002). (h) The scalar potential evolution with time. (g) Scalar potential V verses ϕ . Here $\alpha_1 = \beta_1 = 0.5$, $\eta = 1, \phi_0 = 0, A = \lambda = \beta = \alpha = 0.1$, $\mu = 15$ for $\epsilon = -1$ and -15 for $\epsilon = 1$. The color figure can be viewed online.



(a) $\epsilon = +1$



(b) $\epsilon = -1$



(c) dp/dt

Fig. 4. ECs and sound speed for the hybrid model. Negative sound speed for the phantom field. The color figure can be viewed online.

positive. Figure 4(c) shows that the sound speed causality condition is satisfied only within a specific time interval (for late-times) for a normal scalar field, while it is always violated for the phantom field. The phantom field, for both the hyperbolic and hybrid models, has a positive pressure $p_\phi > 0$ and a negative scalar potential $V(\phi)$. Also, its energy density $\rho_\phi = E_k + V$ takes negative values when the equation of state parameter $\omega_\phi < -1$. Figure 4(b) shows that $p_i + \rho_i \geq 0$ for both normal and phantom fields.

4. CONCLUSION

We revisited the scalar field cosmology in $f(R,T)$ gravity through two models. The main points can be summarized as follows:

- The evolution of the deceleration parameter indicates that a decelerating-accelerating cosmic transit exists in both models . The jerk parameter also tends to 1 at late-times, where the model tends to a flat ΛCDM model.

- The evolution of the varying cosmological constant in both models shows that it tends to a tiny positive value at the current epoch.

- The scalar field pressure p_ϕ in both models shows a sign flipping from positive to negative for a normal scalar field $\epsilon = +1$, but it's always positive for the phantom field $\epsilon = -1$.

- In both models, the scalar potential $V(\phi) > 0$ for $\epsilon = +1$ and < 0 for $\epsilon = -1$.

- For the normal field, $\rho_\phi > 0$ with no crossing to the phantom divide line for ω_ϕ. For the phantom field we have $\rho_\phi < 0$ when $\omega_\phi < -1$.

- Classical energy conditions have been tested for both cases. For the hyperbolic model, the sound speed causality condition $0 \leq \frac{dp}{d\rho} \leq 1$ is valid only for $\epsilon = +1$. For the hybrid model, this condition is satisfied only for a specific interval of time for the normal scalar field.

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