RevMexAA (Serie de Conferencias), **4**, 27–34 (1996)

CHAOS AND THE DYNAMICS OF RESONANT ASTEROIDS

Sylvio Ferraz-Mello[1] and Tatiana A. Michtchenko[1]

RESUMEN

Esta comunicación trata del comportamiento caótico de los asteroides resonantes, tales como ellos son determinados por mapas y simulaciones de muy larga duración. Los resultados muestran que la gravitación es suficiente para haber formado los vacíos de la distribución de los asteroides en las resonancias a través de difusión orbital caótica y esparcimiento de asteroides en órbitas muy excéntricas por encuentros con los planetas.

ABSTRACT

This short review concerns the chaotic behaviour of asteroids in the main resonances as currently determined by maps and simulations over $10^7 - 10^8$ years. Current results confirms that gravitation is sufficient to have formed the gaps in the asteroids distribution at resonances through chaotic orbital diffusion and scattering of asteroids in high-eccentricity orbits by encounters with planets.

Key words: **MINOR PLANETS**

1. CHAOS. CHAOTIC TRANSITIONS

There is a large category of problems in Celestial Mechanics which is characterized by a conservative gravitational evolution for times as long as the age of the Solar System. When dissipative forces may be ruled out for so long times, the system is likely to show chaotic behaviour. The absence of dissipative forces is necessary since usually dissipative forces act driving the system towards almost stationary solutions, in which case chaos is not expected to occur. Thus, planets and asteroids, moving around the Sun in a medium whose density is almost zero, do not dissipate energy. The evolution is entirely determined by exchanges of energy and angular momentum between the bodies and the total energy and angular momentum of the system does not change.

But, what is chaos? Chaos is the behaviour observed in many natural phenomena and is marked by the extreme sensitivity of the solutions to initial conditions. Very close initial conditions lead, after a time, to very different behaviours. People working in critical experiments know that an experiment running fine today may just do not run at all tomorrow! One of the characteristic of chaos is just unpredictability. Consider, for instance, the motion of our planet. Let us consider two Earths, one real and another fictitious. The motion of the Earth is known to be chaotic with an exponential divergence of neighbour orbits of some 1000 times in 40–50 Myrs. Thus, if the 2 Earths are close to each other by 1 meter, in much less than 200 Myr they will be far one from the other by more than their distances to the Sun! Now, the question is: Do we know the position of the Earth with a precision of 1 meter? No, we don't. Then, in the above story, who is the real Earth, who is the fictitious one? We don't know! This just means that we don't know where the Earth will be in 200 Myr. And this is not a mark of our ignorance, but an inherent behaviour of complex gravitational systems. If we improve our knowledge of the present position of the Earth to 1 micron, this only means that instead of 'losing' it in 200 Myr, it will be 'lost' in 300 Myr! This is chaos. The improvement of the knowledge of the initial conditions changes only slightly the duration of the validity of the solution.

Chaotic phenomena in celestial motions were discovered in the years 60. In the beginning, only a few examples were known and, for a while, to find a new case was a 'hot' result. Today, the situation is changed. We know that all celestial motions are, in some extent, chaotic. Conservative non-chaotic motions only exist in textbooks! One well known non-chaotic textbook system is the simple pendulum, an ideal system formed by a rigid wire with a weight in an extremity and having the other extremity tied to an axis by a perfect, frictionless, joint. If an impulsion is given to the ideal pendulum it will oscillate and will remain oscillating

[1]Instituto Astronômico e Geofísico, Universidade de São Paulo, Brazil; sylvio@iagusp.usp.br.

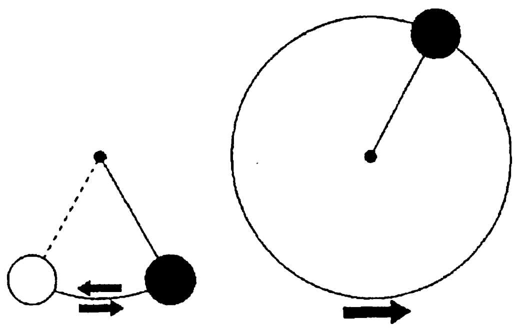

Fig. 1. Regimes of motion of a pendulum.
(a) Oscillation. (b) Rotation.

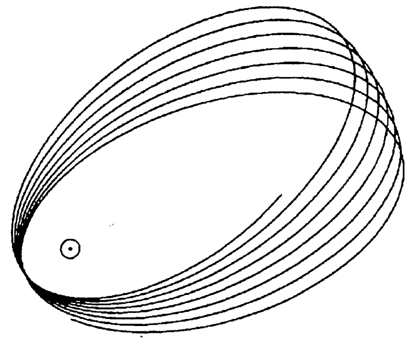

Fig. 2. Moving Keplerian ellipse.

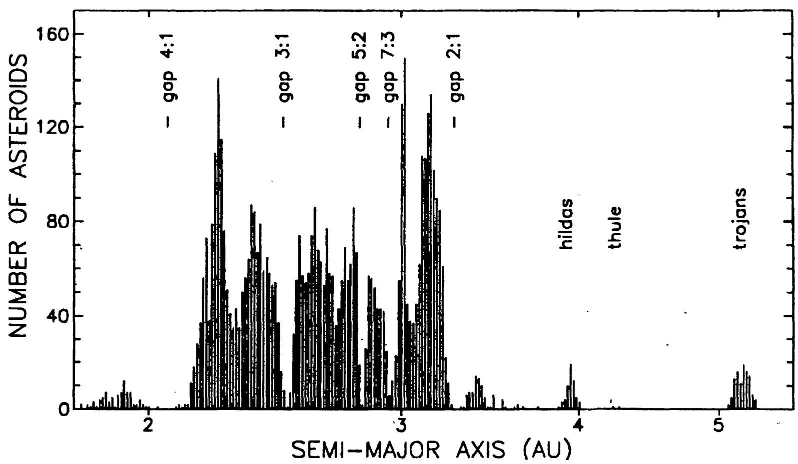

Fig. 3 . Histogram of the number of asteroids vs. semi-major axis showing the location of the main gaps and groups (from Ferraz-Mello 1994a).

indefinitely, always in the same way. The oscillation amplitude will be larger or smaller according with the energy communicated to the pendulum by the initial impulsion. However, if the initial impulsion is larger than a critical minimum, instead of oscillating, the pendulum will rotate, again indefinitely. Thus, the ideal pendulum is a dynamical system with two regimes of motion: oscillation and rotation. These two regimes are perfectly separated and given one impulsion the pendulum will either oscillate or rotate. If, however, our pendulum is cast in iron and is put to move near a variable magnet, the absolute separation of the two regimes of motions ceases to exist. Given an impulsion the pendulum may oscillate, rotate, or have a motion alternating from rotation to oscillation and vice versa. This also is chaos! Solutions close to the critical separation between oscillation and rotation will be extremely sensitive to initial conditions.

The alternation of regimes of motion is one characteristic of planetary motions. However, do not expect that one planet or asteroid may stop his motion and starts moving backwards! But the planetary orbits are moving ellipses, and these ellipses may do it. The ellipse in which is moving an asteroid, for instance, may either rotate or oscillate and in critical situations may change of regime of motion many times.

2. RESONANT ASTEROIDS

Several features in the distribution of the asteroids are associated with chaos and resonance. They are, for instance, the gaps appearing in plots where the horizontal axis displays the semi-major axis or the mean motion of their orbits. Their existence was discovered by Kirkwood, in 1867, and they are known as Kirkwood gaps.

They are located at positions corresponding to the 4/1, 3/1, 5/2, 7/3 and 2/1 commensurabilities between the orbital periods of Jupiter and of the asteroid.

Throughout the 20^{th} century, several theories have been formulated to explain the Kirkwood gaps. One of them, the statistical theory, claims that the gaps are only apparent: asteroids near resonances oscillate about the exact resonance value and expend most of the time far from the commensurabilities. This motion, known as libration, or σ-libration, is characterized by the oscillation, around a fixed value, of the angle

$$\sigma = (r+1)\lambda_{Jup} - r\lambda - \varpi ,$$

(λ_{Jup} is Jupiter's mean longitude, λ and ϖ are, respectively, the asteroid's mean longitude and longitude of the perihelion, r is a rational number and $(r+1)/r$ is the commensurability ratio between the orbital periods of Jupiter and the asteroid). The existence of the libration is known since the early work of Poincaré and notwithstanding having, for some asteroids, a very large amplitude (up to 0.15 AU, see Ferraz-Mello 1988), it is not large enough to originate a statistical gap as wide as those observed in the actual asteroidal distribution. Moreover, current numerical simulations over 10^8 years do not show any oscillation large enough to produce such an effect. On the contrary, they show that in the observed gaps, asteroids librating about the resonance are exactly the ones that are missing.

All other theories assume that the gaps are real and either primordial or the result of the orbital evolution of the asteroids. Gravitational theories say that pure gravitational evolution is sufficient to explain the gaps. It is difficult to have a direct confirmation of these theories, even using numerical simulations, because of the large time interval elapsed since the origin of the asteroidal belt. However, they stand high in favor since Wisdom's work on the formation of the gap in the 3/1 resonance. Wisdom (1982, 1983) showed that the chaotic diffusion of these orbits strongly affects their eccentricities and is able to drive an asteroid to orbits approaching Mars closely in a short time-scale ($10^5 - 10^6$ years). Later results, using more complete models, pointed out that these asteroids can, in fact, be driven to orbits diving deep inside the inner Solar System and, even, coming close to the Sun.

Cosmogonic theories assume that the gaps were formed in the early stages of formation of the Solar System. They assemble assumptions on the processes at work during the formation of the Solar System, in some primordial scenario able to produce gaps in the early asteroidal distribution. To the extent that the gravitational theory becomes widely accepted as a general explanation for the scattering of asteroids in resonances, cosmogonic hypotheses becomes less important. Moreover, if the efficiency of gravitational evolution mechanisms is as large as pointed out by some recent investigations, the present state of the asteroidal belt at resonances depends only weakly on its primordial state and cannot provide a significant information on the processes prevailing at the belt formation.

3. THE 3/1 RESONANCE

Until 1982, the only mode of motion known in this resonance was the ordinary low-eccentricity regime where, typically, the perihelion retrogrades and the eccentricity has a small periodic variation (see **a** in Fig. 4 right). Some numerical experiments had shown anomalous increases of the eccentricity (Scholl & Froeschlé 1974), but the dynamics of this resonance was not unravelled until Wisdom (1982, 1983) discovered the existence of a mid-eccentricity mode of motion, in which the asteroid's perihelion oscillates about the position of Jupiter's perihelion and the eccentricity has large oscillations approaching values as high as 0.4 (see **b** in Fig. 4 right). Wisdom also showed that orbits having a seemingly regular low-eccentricity motion for long times and suddenly transit to the mid-eccentricity regime are common. Generic orbits alternate between these two modes of motion in a short timescale (much less than 1 Myr). From the cosmogonic point of view, the important fact is that, in the mid-eccentricity regime, the asteroid orbit crosses the orbit of Mars and the asteroid may, eventually, have a close approach to the planet. When this happens, the orbit suffers an important energy change and leaves the resonance. The possibility of this scattering could explain the observed absence of permanent asteroids in the 3/1 resonance. The only doubts concerning this scenario come from the fact that Mars is a small planet and we do not know if this process could have been efficient enough to expel all asteroids expected to be there after the formation of the asteroid belt, 4.5 billion years ago. Later on, Ferraz-Mello & Klafke (1991) introduced a new theoretical model allowing them to extend the analysis high eccentricities. They have shown the existence of a very-high-eccentricity regime where the eccentricity oscillates in a wide range reaching values close to 1 and the perihelion may perform a complete revolution before the eccentricity decreases again (see **c** in Fig. 4 right).
In the energy range considered by Ferraz-Mello & Klafke, this regime is almost always separated from the other ones by some regular motions (Fig. 4 left) able to avoid transitions between them, at least in a timescale as short as the one observed in the transitions between the regimes **a** and **b**. However, decreasing the energy, these

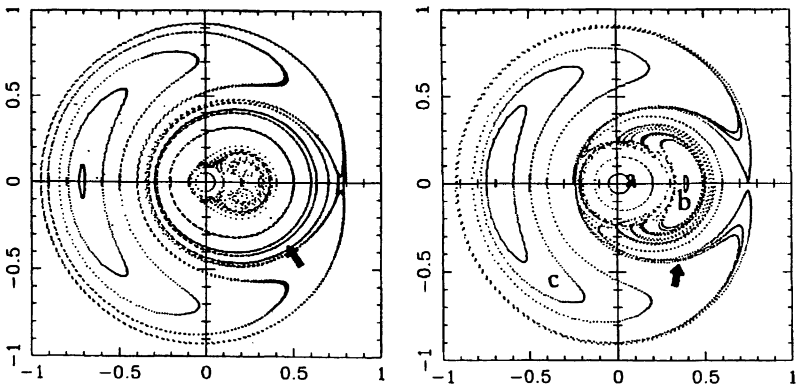

Fig. 4. Poincaré maps ($\sigma = \pi/2$, $\dot{\sigma} < 0$) of the resonance 3/1 in the frame of the planar averaged Sun-Jupiter-asteroid problem at two different energy levels. Coordinates are $x = e.\cos(\varpi - \varpi_{\rm Jup})$, $y = e.\sin(\varpi - \varpi_{\rm Jup})$. On the left side, the chaotic domain found by Wisdom is seen in the inner part of the figure (the whole set is the plot of one solution). It is confined by a bunch of almost regular motions (isolated curves; see the arrow). On the right side, these regular motions no more exist and a heteroclinic bridge (arrow) allows transitions to the high-eccentricity regime of motion (after Ferraz-Mello & Klafke 1991).

seemingly regular orbits cease to exist and a heteroclinic bridge appears allowing the solutions to go from **b** to **c** (Fig. 4 right) and the eccentricity to grow to values close to 1. Decreasing the energy still more, the modes of motion studied by Wisdom become parted and direct transitions between low and high-eccentricities become possible (Klafke et al. 1992). The intermittences involving this new mode of motion can drive the asteroidal eccentricity to values close to 1 and back. In this case, the asteroid will not only cross the orbit of Mars, but also those of the Earth and Venus, planets which are 10 times more massive than Mars. When the effects of all outer planets are considered, excursions to very high eccentricity are the rule (see Moons & Morbidelli 1995). Farinella et al. (1993) and Morbidelli & Moons (1995) have shown orbits whose eccentricity is high enough to allow the asteroid to fall in the Sun. In fact, the asteroid becomes liable to be disrupted and transformed into meteoroids even if the eccentricity is not so close to 1 to allow it to reach the Sun.

4. THE 2/1 RESONANCE

The analytical study of the 2/1 and 3/2 resonances is impaired by the small convergence radius of the disturbing function expansions. One way to study the dynamics of this resonance is to perform numerical integrations and to smooth the output by filtering out the high frequencies (Michtchenko 1993; Michtchenko & Ferraz-Mello 1995). When the model is the planar Sun-Jupiter-asteroid model, these smoothed outputs may be interpreted as solutions of an averaged dynamical system with 2 degrees of freedom and allow the construction of Poincaré maps (Ferraz-Mello 1994b). Figure 5 shows some of these maps obtained from 1 Myr numerical integrations. In the maps, the motion regimes denoted **a** and **b** in the 3/1 resonance are almost absent; they are only seen in Poincaré maps corresponding to orbits with very large libration amplitudes ($\Delta\sigma > 200^o$) (see Fig. 5 *right*). Generally, these regimes are substituted by a single chaotic low-eccentricity zone seen in the central part of the map (see Fig. 5 *middle*) (Giffen 1973; Froeschlé & Scholl 1981). Lemaître & Henrard (1990) have shown that this chaotic zone has its origin in the existence of resonances (said secondary) between the libration of the critical angle σ and the perihelion motion. There is, in fact, a succession of microregimes of motion corresponding to these secondary resonances whose overlaps allow an orbit to transit through them. This chaotic zone is confined to low-eccentricities by apparent regular motions visible between $e \sim 0.2$ and $e \sim 0.5$ and is only slightly affected when more complete models are used (Michtchenko & Ferraz-Mello 1996). However, when these apparently regular regions are studied with the more accurate tools of wavelet and frequency analyses, the chaoticity associated with secondary resonances of higher orders becomes apparent and, when Saturn is added to the model, transitions between neighbouring secondary resonances occur (Michtchenko & Nesvorný 1996; Nesvorný & Ferraz-Mello 1996).

The high-eccentricity mode of motion seen in the 3/1 resonance has, in the 2/1 resonance, two counterparts:

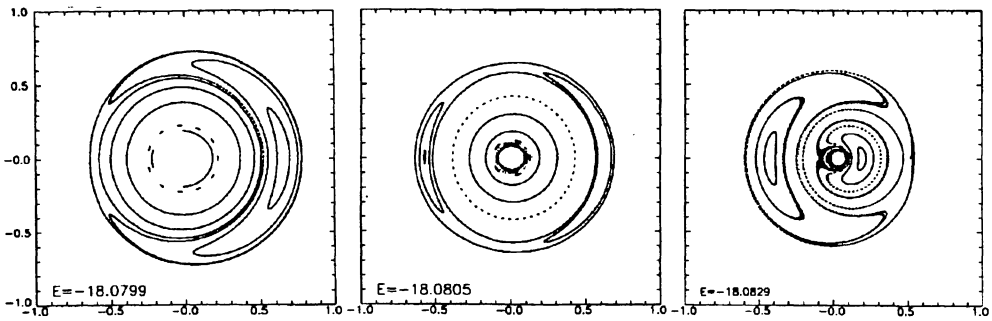

Fig. 5. Poincaré maps ($\sigma = 0$, $\dot{\sigma} > 0$) of the resonance 2/1 in the frame of the planar averaged Sun-Jupiter-asteroid problem. Coordinates as in Fig. 4. (from Ferraz-Mello 1994b)

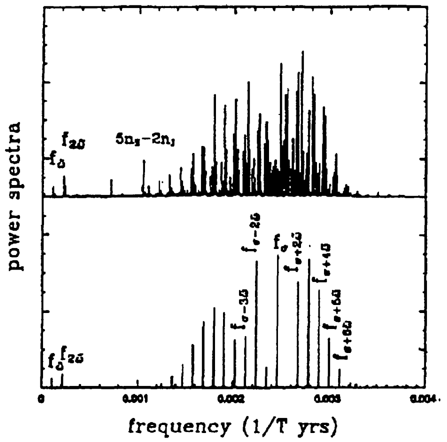

Fig. 6. FFT power spectra of solutions starting at $e_0 = 0.3$ in the frame of two different models. *Bottom*: Sun-Jupiter-asteroid. *Top*: Sun-Jupiter-Saturn-asteroid.

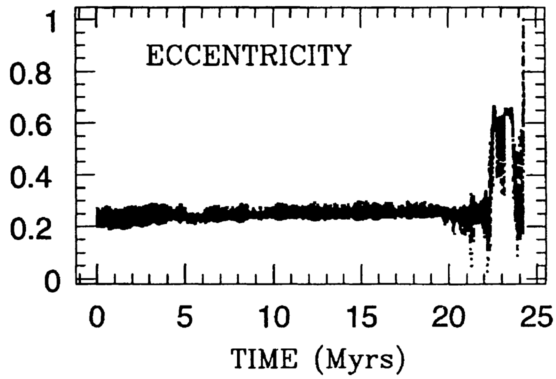

Fig. 7. Variation of an asteroid eccentricity showing $e \to 1$ in 25 million years.

One on the left side of the Poincaré map (asteroid's perihelion librating about Jupiter's aphelion) and another on the right side (asteroid's perihelion librating about Jupiter's perihelion). The motion in these regimes is such that both the critical angle σ and the perihelion are librating.

The separation between the high-eccentricity lobes and the low-eccentricity chaotic zone persists even when the long-period perturbations of the orbit of Jupiter are considered (Morbidelli & Moons 1993). The robustness of this separation led many authors to find impossible to explain the Kirkwood 2/1 gap by chaotic diffusion followed of gravitational scattering, as for the 3/1 gap. However, when the action of Saturn is also considered, all solutions become clearly chaotic. This has been first shown by means of the calculation of the maximum Lyapunov exponents (Ferraz-Mello 1994b) and later on confirmed by Fourier spectral analyses. The spectrum of one solution of the Sun-Jupiter-asteroid model is shown in Fig. 6 *bottom*. The well-defined spectral lines associated with the independent modes of motion indicate regular motion. The spectrum of the analogous orbit calculated with inclusion of the action of Saturn (Fig. 6 *top*) shows many additional spectral lines associated with the secular modes of motion and is characteristic of chaotic motion and time variation of the proper frequencies. These results show that Saturn triggers the destruction of the almost regular structures separating the low-eccentricity chaotic region and the high-eccentricity lobes. More recently, Henrard et al. (1995) explained the origin of this stochasticity by intermittent mechanisms associated with secular resonances between the node of the orbits of the asteroid and Jupiter as well as resonances of the argument of the perihelion (ω). The Hecuba gap is the cosmogonic consequence of this global stochasticity. We have done some tens of exploratory numerical integrations of the exact equations in the central part of the resonance. Most of the results show a jump in eccentricity to 0.9, or more, in a time from 15 to 150 Myr, followed by the escape from the resonance (Fig. 7). These integrations confirm the stochasticity of the 2/1 asteroidal resonance and its role in the formation of the Hecuba gap.

5. THE 3/2 RESONANCE

In this resonance, the asteroids distribution, instead of showing a gap as in previous cases, shows an isolated accumulation of asteroids, in the very depleted zone external to the main belt. 58 asteroids are currently known in this resonance. Its paradigm —(153)Hilda— was discovered by J. Palisa, an active asteroid discoverer that discovered 83 of the 323 asteroids known by 1891. As a group, the *Hildas* are characterized by limited range of the proper eccentricities and inclinations: $0.1 < e < 0.3$ and $I < 20°$. These particularities of the distribution shape of the *Hildas* may be explained in the context of the pure gravitational evolution.

The 3/2 resonance was studied using the same techniques used to study the 2/1 resonance. Figure 8 shows two Poincaré maps of this resonance. They differ from those of Fig.5 in several aspects. For instance, we devise only two regimes of motion:(a) the inner domain of perihelion circulation and (b) the perihelion libration lobe on the left side. At variance with the 2/1 resonance, no apparent chaotic activity exists in the region of perihelion circulation (confirmed by Lyapunov exponents tending to zero in numerical integrations over 10 Myr); The analysis of these solutions shows the same kind of secondary resonances web responsible for the inner chaotic region in the 2/1 resonance, but the chaotic regions associated with every secondary resonance in the web seem to be very narrow and they do not overlap (Michtchenko 1993; Michtchenko & Ferraz-Mello 1995). The only source of appreciable chaoticity is the bifurcation between the two modes of oscillation of the perihelion. Strong chaos is visible spreading itself over a large part of the perihelion libration lobe and in the outer region around the two regimes. The non-existence of observed asteroids with mean orbital eccentricities larger than ~ 0.3 is due to the fact that the outer orbits are scattered by approaches to Jupiter itself. The inclusion of Saturn in the models accelerates this phenomenon allowing the orbits to be scattered in less than 1 Myr. The results of Morbidelli and Moons (1993) for this resonance also show an extended chaotic region surrounding seemingly regular motions with $e < 0.25$.

The spectral analysis of the output of numerical integrations shows a complex of chaotic orbits with eccentricities lower than 0.1, associated with secondary resonances involving the libration frequency f_σ and twice the frequency of the argument of the perihelion motion f_ω ($\omega = \varpi - \Omega$) (Michtchenko & Ferraz-Mello 1996). The rapid transitions between these resonances drive the orbit away from this region and thus explains the non-existence of orbits with small e among the Hildas.

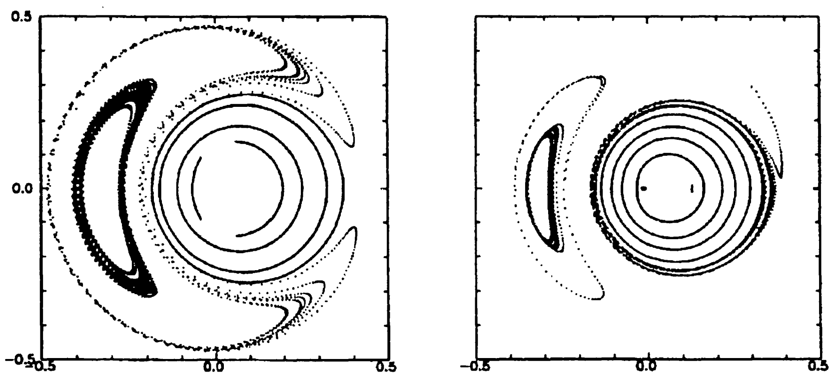

Fig. 8. Poincaré maps ($\sigma = 0$, $\dot{\sigma} > 0$) of the 3/2 resonance in the frame of the planar averaged Sun-Jupiter-asteroid problem. Coordinates as in Fig. 4. Orbits in the perihelion libration lobe are highly chaotic and are bound to close approaches to Jupiter in short times. Orbits in the innermost part remain regular for long times even when inclinations and perturbations of Saturn are taken into account. The actual Hildas are in the inner region (from Ferraz-Mello 1994a).

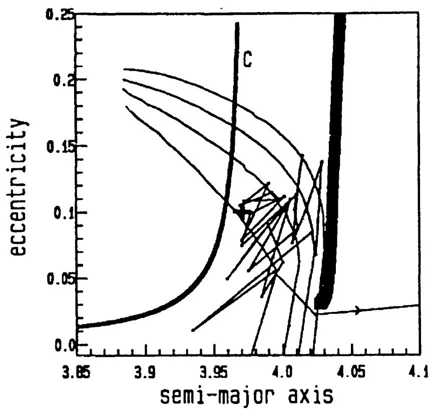

Fig. 9. Chaotic diffusion of a solution through the secondary resonances $f_\sigma/f_{2\omega}$ from 1/1 to 1/4 in the 3/2 resonance. The initial position of the asteroid is shown by a cross.

As an example, we show the chaotic diffusion of an orbit with initial conditions deep inside the region of overlap of secondary resonances of the type f_σ/f_ω. Figure 9 shows the (a,e)-plane for $I_0 = 15°$ and the initial position of the orbit is marked by a cross. The curves inside the resonance region indicate the loci of the secondary resonances from 1/1 to 1/4. The rapid chaotic transition across the overlapping zone produces the slow diffusion of the orbit along a band of overlapping resonances and the escape after 10 Myrs, when the orbit approaches the resonance border indicated by the thick curve. The few existing low-eccentricity *Hildas* are outside this overlapping zone. Our experiments also showed that the zone of overlap of these secondary resonances grows with the increase of the inclination and for $I > 20°$ covers the whole phase space of the 3/2 resonance; this fact explains that there are no members of the Hilda group with high inclination. When the perturbations due to Saturn are taken into account and the asteroid is left to move in an inclined orbit, the maximum Lyapunov exponents obtained following the inner regular orbits still tend to very small values. The values found are in the range $10^{-5.5} - 10^{-7}\mathrm{yr}^{-1}$.

The cosmogonic implication of these small values is that the chaotic processes at work in the domain where the Hildas are found, the same acting in the 2/1 resonance, are, now, about one hundred times slower. We may say that the same process responsible for the depletion of the 2/1 resonance are depleting the 3/2 resonance, but at a slower pace and the time required to complete this depletion is some orders of magnitude larger than the Solar System age. These values are very small and coherent with the observed existence of about 60 numbered asteroids in this resonance.

6. CONCLUSION

We reviewed the evolution of 3/1, 2/1 and 3/2 resonant asteroids, emphasizing the several regimes of motion existing in each case and the chaotic transitions between these modes. The collected results refer to current maps and simulations extending over $10^6 - 10^8$ years. They show that several conclusions, obtained in the past on the basis of simulations over $10^4 - 10^5$ years, were not correct. In the same way, we cannot assume the current conclusions as definitive. They are certainly an improvement on the previous scenario but we do not know what will be unraveled when our theories become able to show evolutions over as large as 10^9 years. It was shown that asteroids in the resonances 3/1 and 2/1 may have huge variations in the eccentricity. At variance with these resonances, the chaotic transitions in the 3/2 resonance may be so slow that the necessary timespan to unravel the main dynamical mechanisms at work in this resonance are larger than the age of the Solar System. These results are coherent with the high depletion observed in the 2/1 and 3/2 resonances and with the existence of almost 60 known asteroids in the 3/2 resonance.

This investigation was sponsored by the Research Foundation of São Paulo State (FAPESP).

REFERENCES

Farinella, P., Froeschlé, Ch. & Gonczi, R. 1993, Celest. Mech. Dyn. Astron. 56, 287
Ferraz-Mello, S. 1988, AJ 96, 400
Ferraz-Mello, S. 1994a, in Asteroids, Comets, Meteors 1993, ed. A.Milani, M.DiMartino & A.Cellino (eds.), IAU Symp. 160, (Dordrecht: Kluwer), 175
Ferraz-Mello, S. 1994b, AJ 108, 2330
Ferraz-Mello, S., Dvorak, R. & Michtchenko, T.A. 1995, in From Newton to Chaos, ed. A.E.Roy & B.A.Steves, (New York: Plenum Press), 157
Ferraz-Mello, S. & Klafke, J.C. 1991, in Predictability, Stability and Chaos in N-Body Dynamical Systems, ed. A.E.Roy, (New York: Plenum Press), 177
Froeschlé, C. & Scholl, H. 1981, A&A 93, 62
Giffen, R. 1973, A&A 23, 387
Henrard, J., Watanabe, N. & Moons, M. 1995, Icarus 115, 336
Klafke, J.C., Ferraz-Mello, S. & Michtchenko, T. 1992, in Chaos, Resonance and Collective Dynamical Phenomena in the Solar System, ed. S.Ferraz-Mello, (Dordrecht: Kluwer), 153
Lemaître, A. & Henrard, J. 1990, Icarus, 83, 391
Michtchenko, T.A. 1993, Doctorate Thesis, University of São Paulo
Michtchenko, T.A. & Ferraz-Mello, S. 1995, A&A 303, 945
Michtchenko, T.A. & Ferraz-Mello, S. 1996, A&A, (in press)
Michtchenko, T.A. & Nesvorný, D. 1996, A&A, in press
Moons, M. & Morbidelli, A. 1995, Icarus, 114, 33
Morbidelli, A. & Moons, M. 1993, Icarus, 102, 316
Morbidelli, A. & Moons, M. 1995, Icarus, 115, 60
Nesvorný, D. & Ferraz-Mello, S. 1996, A&A, submitted
Scholl, H. & Froeschlé, C. 1974, A&A 33, 455
Wisdom, J. 1982, AJ 85, 1122
Wisdom, J. 1983, Icarus 56, 51