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# THE RIEMANN FRAMEWORK AND HIGHER ORDER GODUNOV SCHEMES FOR PARALLEL, SELF-ADAPTIVE COMPUTATIONAL ASTROPHYSICS

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RESUMEN

El campo de la astrofísica computacional (CA) ha tenido muchos avances recientes. La aparición de esquemas de orden mayor de Godunov para varios de los sistemas computacionales de interés astrofísico representa un gran avance en el campo. Estos esquemas permiten mayor exactitud y menor disipación. El hecho de que estos esquemas estén fundamentados en las características físicas asociadas a las ecuaciones hiperbólicas, les permiten una mayor fidelidad de los fenómenos físicos comparado con las formulaciones computacionales anteriores. Estos métodos han sido extendidos a MHD clásica y relativista, hidrodinámica relativista y a MHD e hidrodinámica radiativa. Una ventaja adicional es que los esquemas se pueden paralelizar y aceptan estrategias auto-adaptivas. Varios de los métodos han sido desarrollados por el autor y sus colaboradores, en un esfuerzo por aumentar la calidad de la CA y aquí se describe el trabajo. Los métodos se han implementado en una plataforma RIEMANN para CA altamente paralelizable y auto-adaptiva.

ABSTRACT

The field of computational astrophysics (CA) has seen many recent advances. The emergence of higher order Godunov schemes for many of the systems of interest in CA represents a development of great importance in this field. Such schemes offer high order accuracy and low dissipation. The fact that such schemes derive their underpinnings from physical features in the hyperbolic equations results in their having greater physical fidelity and reliability compared to older formulations. These methods have been shown to be extensible to non-relativistic MHD, relativistic hydrodynamics, relativistic MHD, radiation hydrodynamics and radiation MHD. A further advantage is that such schemes also take well to parallel, self-adaptive strategies for their solution. Many of these methods have been developed by this author and his co-workers in an effort to raise the quality of CA simulations and we describe that work here. The methods have been implemented in the RIEMANN framework for highly parallel, self-adaptive CA.

*Key Words:* **HYDRODYNAMICS — METHODS: NUMERICAL — MHD**

## 1. INTRODUCTION

It is interesting to view this conference as a microcosmic indicator of trends in astronomy, with numerous advances having been reported in theory and observation. Even more interesting is the growing trend amongst astronomers to use techniques drawn from computational astrophysics (CA) in their work. However, a quiet revolution has been taking place in CA. In the eighties the ability to calculate fluid flow with fidelity using higher order Godunov schemes caused an interesting revolution in the aerospace community. Suddenly, every aerospace engineer could reliably calculate just about every flow problem s/he could imagine. The new higher order Godunov codes easily displaced the earlier artificial viscosity-based codes in aerospace. In the nineties, a few computational astrophysicists developed higher order Godunov schemes for a range of hyperbolic systems that include MHD, relativistic hydro, relativistic MHD, radiation hydrodynamics and radiation MHD. Thus the same paradigm shift that caused a revolution in aerospace now becomes possible in CA. It is hoped that astronomers will see the virtue of replacing the artificial viscosity-based codes that are currently in use in CA.

Matching this advance in technique is a change in technology. The emergence of parallel computers and their easy availability means that problems in CA can be carried out with unprecedented resolution, thus enhancing their reliability. The higher order Godunov schemes offer a further advantage in that they are extremely cache-friendly for the emerging architectures and parallelize extremely well.

Any observation of astrophysical phenomena shows that they take place over a range of length scales. This necessitates simulating them on multiple scales. Fortunately, adaptive mesh refinement (AMR) techniques have been developed which permit such a multiscale viewpoint. The ability to do AMR simulations on parallel machines has been something of a challenge until recently. The author's RIEMANN framework for CA provides a general approach for solving parallel self-adaptive problems in hydrodynamics, MHD, relativistic hydrodynamics, relativistic MHD, radiation hydro and MHD. These systems are solved using some of the most sophisticated higher order Godunov techniques to date, developed by the author and collaborators.

Section 2 gives a brief introduction to higher order Godunov schemes. In § 3 we catalogue the techniques that have been developed for various hyperbolic systems. In § 4 we discuss the RIEMANN framework for parallel, self-adaptive CA. In § 5 we draw some conclusions.

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## 2. INTRODUCTION TO HIGHER ORDER GODUNOV SCHEMES

In this section, we give the briefest possible introduction to higher order Godunov schemes for solving systems of hyperbolic equations. Two reviews that are geared towards CA are those of LeVeque (1997), focusing on theory, and Balsara (1997), focusing on practice. Since then, many more systems of equations of interest to CA have come within the fold of higher order Godunov schemes. Many of these systems can be written in the form

$$\partial_t U + \partial_x F(U) + \partial_y G(U) + \partial_z H(U) = 0 \quad , \tag{1}$$

where $U$ is a vector of $n$ conserved variables and $F$, $G$ and $H$ are fluxes in the $x$, $y$ and $z$-directions. Such equations often admit wave-like solutions in the absence of dissipative processes. Thus, one can conceptualize the time-evolution of such systems as the evolution *and interaction* of waves. Usually, it is simpler to demonstrate the wave-like structure of such equations when they are written out in terms of an alternate set of variables that are referred to as the primitive variables. We denote the primitive variables by $V$. Thus the $x$-directional variation of the above equations in terms of the primitive variables can be written as:

$$\partial_t V + A(V)\partial_x V = 0 \quad , \tag{2}$$

where $A(V)$ is an $n \times n$ matrix. The system is hyperbolic if $A(V)$ has $n$ real eigenvalues corresponding to the propagation of $n$ wave-like structures. Corresponding to the eigenvalues, there are right eigenvectors which give us information about the structure of the waves. For each right eigenvector there is a corresponding left eigenvector which, if properly orthonormalized, can be used to give us the strength of an arbitrary fluctuation in terms of the corresponding wave field. When these fluctuations are small, they evolve linearly. However, astrophysical flows often produce large fluctuations in the flow variables. In such situations, the evolution can be strongly non-linear and has to be developed in terms of the Riemann problem, which gives us the structure of the waves that emerge from two discontinuous slabs of fluid that are initially put side by side.

Godunov (1959) found a way of solving systems of hyperbolic equations that respects and incorporates this wave-like structure. His idea was to discretize the fluid flow into slabs of fluid. One slab of fluid was ascribed to each zone in the computational problem. The discontinuities between zones were allowed to evolve using the Riemann problem. The flow variables that emerged in the Riemann problem were used to compute the fluxes in equation 1. The fluxes were then used to obtain the time-update. The very attractive aspect of the method was that since it was built on physical building blocks (the Riemann problem), it always yielded the correct physical answer. However, the original Godunov scheme suffered from the fact that it was very dissipative because of its low order of accuracy.

Van Leer (1979) overcame the limitations of Godunov's scheme by giving the slabs a linear variation of the flow variables within each zone, thus improving accuracy. The extent of this variation was moderated so as to ensure numerical stability. The resulting higher order Godunov schemes were further improved by a number of authors. Roe (1981) designed linearized Riemann solvers which were very computationally efficient. Harten (1982) showed that interpolating in the characteristic variables improves the solution quality and he formulated total variation diminishing (TVD) schemes which bound the variation in these variables. Harten et al. (1987) formulated essentially non-oscillatory (ENO) schemes with higher order accuracy than TVD schemes. Shu & Osher (1988, 1989) showed that such ENO schemes could be made more efficient. Recent improvements in the order of accuracy have come from Liu, Osher, & Chan (1994), Jiang & Shu (1996), and Balsara & Shu (2000).

## 3. HYPERBOLIC SYSTEMS OF INTEREST TO COMPUTATIONAL ASTROPHYSICS

**MHD:** Jeffery & Taniuti (1964) give a very good analysis of the structure of MHD waves and shocks. An analysis of the eigenstructure of MHD with an eye to computation was done by Zachary & Colella (1992), and done with higher precision by Roe & Balsara (1996). They showed that an orthonormal and complete set of left and right eigenvectors always exists. Thus a loss of strict hyperbolicity does not cause a computational breakdown. They also showed that the flux computation based on the linearization of the hyperbolic system remains valid even at the triple-umbilic point. Falle & Komissarov (1997) and Myong (1996) have also worked on the issue of compound waves, first found to be present in higher order Godunov codes by Brio & Wu (1988). The upshot is that the codes are picking up the solution that they should.

Dai & Woodward (1994a) developed a non-linear Riemann solver for numerical MHD. Such Riemann solvers are extremely floating point intensive. The current focus is on the development of linearized Riemann solvers (Balsara 1998a, 1999e), which give high-quality results at a much lower operation count.

TVD schemes for MHD have been designed by Zachary, Malagoli, & Colella (1994), Ryu & Jones (1995) (who also give an excellent set of MHD test problems), and Balsara (1998b). Test problems allow the differences in quality between MHD codes to be made apparent. Dai & Woodward (1994b) have designed PPM (piecewise parabolic method) schemes for MHD. The advances in ENO schemes have also led to the design of very high order schemes for MHD (Jiang & Wu 1999; Balsara & Shu 2000).

Magnetic fields satisfy the constraint that they remain divergence-free over the course of their evolution. In the past, some authors, (e.g., see Powell 1994), have changed the structure of the equations of MHD so that the magnetic fields can develop a non-zero divergence. More recently, divergence-free evolution of the magnetic fields has been preferred (Balsara & Spicer 1999; Dai & Woodward 1998; Ryu et al. 1998). Balsara & Spicer (1999) showed that the fluxes can be multidimensionally upwinded in response to flow features.

**Relativistic Hydrodynamics:** Balsara (1994) and Martí, Müller, & Ibáñez (1994) designed solution strategies for the Riemann problem for relativistic hydrodynamics. Eulderink & Mellema (1994) and Falle & Komissarov (1996) have developed linearized Riemann solvers for this problem. The introduction of relativistic effects does not change the wave structure compared to that for non-relativistic hydrodynamics. The same TVD and ENO interpolation strategies that work well for non-relativistic hydrodynamics and MHD also work well for relativistic hydrodynamics. There is a slight difference in the equations of relativistic hydrodynamics which stems from the fact that the primitive variables cannot be easily derived from the conserved variables: they require the solution of a transcendental equation. This is one of the finer points which makes relativistic hydrodynamics rather computationally costly in comparison to non-relativistic hydrodynamics.

**Relativistic MHD:** The relativistic analogue of the equations of non-relativistic MHD also exist (see Anlies 1989 and Lichnerowicz 1967 for good introductions to relativistic MHD). A thorough analysis of the eigenstructure of relativistic MHD (Anlies & Pennisi 1987) shows that the degeneracies follow those of the non-relativistic case. TVD schemes and linearized Riemann solvers can be designed for relativistic MHD in much the same way that they can be designed for non-relativistic MHD (Balsara 2000). Again, the primitive variables cannot be easily derived from the conserved variables. In the case of relativistic MHD this necessitates solving not just one but rather a whole set of non-linear transcendental equations.

**Radiation Hydrodynamics:** The analysis of the equations of radiation hydrodynamics (see, e.g., Mihalas & Mihalas 1984), was shown by Balsara (1999a) to admit the usual sound waves, shear waves and entropy wave from hydrodynamics, but with the interesting twist that these waves carry important additional parts

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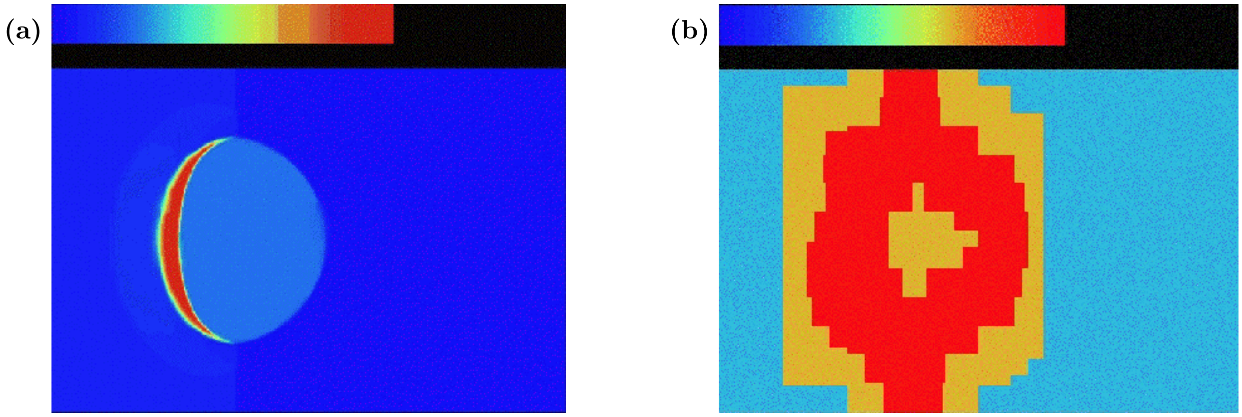


Fig. 1. *(a)* Density and *(b)* grid levels from a self-adaptive MHD shocked cloud problem.

associated with the radiation energy and the radiation fluxes. This shows that the equations of radiation hydrodynamics are more intimately coupled than had previously been thought possible. Furthermore, the system of radiation hydrodynamics also admits a pair of waves, which propagate the radiation energy density, that can travel at luminal speeds consistent with the fact that radiation energy can propagate at the speed of light, and another pair that propagates the radiative flux. As a result, the radiation energy density waves do not produce material fluctuations. The linearized Riemann solver for radiation hydrodynamics has been designed by Balsara (1999b), and Balsara (1999c) analyzes the design of implicit schemes for radiation hydrodynamics.

**Radiation MHD:** The equations of radiation MHD also form a hyperbolic system (Balsara 1999d), and analogues of the fast, Alfvén, slow, and entropy waves in MHD exist even for this system. Again, these waves couple strongly to the radiation energy and radiation fluxes and, as in the above case, the system admits a pair of waves that propagate the radiation energy density and another pair that propagates the radiative flux. The linearized Riemann solver for radiation hydrodynamics has also been designed by Balsara (1999e).

## 4. THE RIEMANN FRAMEWORK

The AMR techniques were originally developed by Brandt (1977), Berger & Oliger (1984) and Berger & Colella (1989). Ever since the advent of parallel supercomputers, it has been a topic of great interest to be able to do parallel AMR. Balsara & Norton (2000) showed that parallel AMR computations can be carried out efficiently in a standard-conforming way. They also showed that frameworked approaches for parallel, self-adaptive computational astrophysics, as exemplified by the RIEMANN framework, could be built. In future work, it will be shown that a divergence-free extension of the strategy of Balsara & Spicer (1999) can be made so that one can compute magnetic fields on AMR hierarchies in a divergence-free fashion.

As an example, Figure 1*a* (from Balsara & Norton 2000) shows the density from a 3D magnetized shock-cloud interaction problem. Figure 1*b* shows a map of the levels in the AMR hierarchy used for solving this problem. Here mid gray denotes the base grid level, light gray the first level of refinement and dark gray the second level of refinement. Balsara & Norton (2000) gives several further details, including a scalability study for the AMR-MHD problem.

## 5. CONCLUSIONS

We have shown that significant recent advances have been made in computational astrophysics owing to the development of higher order Godunov schemes for many of the interesting hyperbolic systems that are of relevance in this field. We have attempted to show that there are important similarities in the structure of these equations and that the similarities should be used to advantage in developing solution techniques. The emergence of the parallel self-adaptive RIEMANN framework for computational astrophysics makes it possible

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to solve large classes of astrophysical problems with extremely high accuracy and resolution.

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