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# THE TWIN PEAK QPOS IN NEUTRON STAR AND BLACK HOLE SOURCES: WHAT IS EXPLAINED, AND WHAT IS NOT

M. A. Abramowicz,[1,2,3] W. Kluźniak,[2,4] M. Bursa,[3] J. Horák,[3] P. Rebusco,[5] and G. Török[6]

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RESUMEN

Las oscilaciones cuasi periódicas de alta frecuencia (OCP-AF) observadas en el flujo de rayos X de binarias galácticas con agujeros negros y estrellas de neutrones se explican mediante el modelo de resonancias propuesto en 2000 por Kluźniak y Abramowicz. Estas se deben a resonancias no lineales en los modos de oscilación en discos de acreción en el régimen gravitacional de campo fuerte. Hay aún problemas que no han sido resueltos, pero las observaciones recientes de OCP-AF proporcionan evidencia directa de resonancias no lineales. Este artículo da una revisión actualizada (mayo 2006) de lo que ha sido explicado, y de lo que falta aún por entender, dentro del modelo de resonancia.

ABSTRACT

The resonance model proposed in 2000 by Kluźniak and Abramowicz, explains the twin peak, high frequency, quasi periodic oscillations (HF QPOs) observed in the X-ray flux from black hole and neutron stars, as a non-linear resonance in modes of oscillations of accretion flows in strong gravity. While several important physical questions remain unanswered, recent observations of HF QPOs provide direct evidence of non-linear resonances. This article gives an up to date (May 2006) review of what has been explained, and what has not been explained, by the resonance model.

*Key Words:* **ACCRETION, ACCRETION DISKS — BLACK HOLE PHYSICS — HYDRODYNAMICS — STARS: NEUTRON**



## 1. TWIN PEAK QPOS IN BLACK HOLE AND NEUTRON STARS

The Fourier power density spectra of X-ray variability in several neutron star and black hole sources in Galactic low mass X-ray binaries (LMXBs), often reveal pairs of "twin peak" QPOs (Figure 1), with frequencies $\nu_U > \nu_L$ .

Kluźniak & Abramowicz (2000,2001) suggested that the twin peak QPOs are caused by a non-linear resonance between two coupled modes of small amplitude oscillations of accretion flow in strong gravity. This suggestion has been developed, with the help of numerous collaborators[7] into the "twin peak QPO resonance model" (for references see Abramowicz et al. 2005). Two (overlapping) methodologies have been used in developing the model: an "intrinsic, mathematical" approach, and an "external, physical" approach.

The mathematical approach allowed a practical and efficient means of identifying secure results, once the basic physics of the QPOs had been identified. It is based on the theory of small oscillations, and offers a standard, well established, purely mathematical scheme (not always technically easy, however) for an exact description of the "intrinsic" oscillatory properties of QPOs. Examples of the ones that have been already *fully* explained by this approach are given below in Section 1.1. Future developments may only add a more detailed physical interpretation, but the essence of the explanation will stay.

Obviously, however, there are important issues that cannot be resolved by pure mathematics of the theory of small oscillations alone. They have been studied in the context of detailed physical models of accretion flows. Some of these physical issues, "external" to the mathematical properties of coupled

[1]Department of Physics, Göteborg University, Sweden.
[2]Copernicus Astronomical Centre, Poland.
[3]Astronomical Institute, Czech Academy of Sciences, Czech Republic.
[4]Institute of Astronomy, Zielona Góra University, Poland.
[5]Max-Planck-Institute for Astrophysics, Germany.
[6]Institute of Physics, Silesian University, Czech Republic.
[7]Didier Barret, Omer Blaes, Axel Brandenburg, Tomasz Bulik, Michal Bursa, Jiří Horák, Vladímir Karas, Shoji Kato, Jean-Pierre Lasota, William Lee, Mami Machida, Jeff McClintock, Jean-Francois Olive, Paola Rebusco, Ron Remillard, Ed Spiegel, Eva Šramková, Nick Stergioulas, Ulf Torkelsson, Gabriel Török, Roberto Vio, Brian Warner, and Zdeněk Stuchlík.

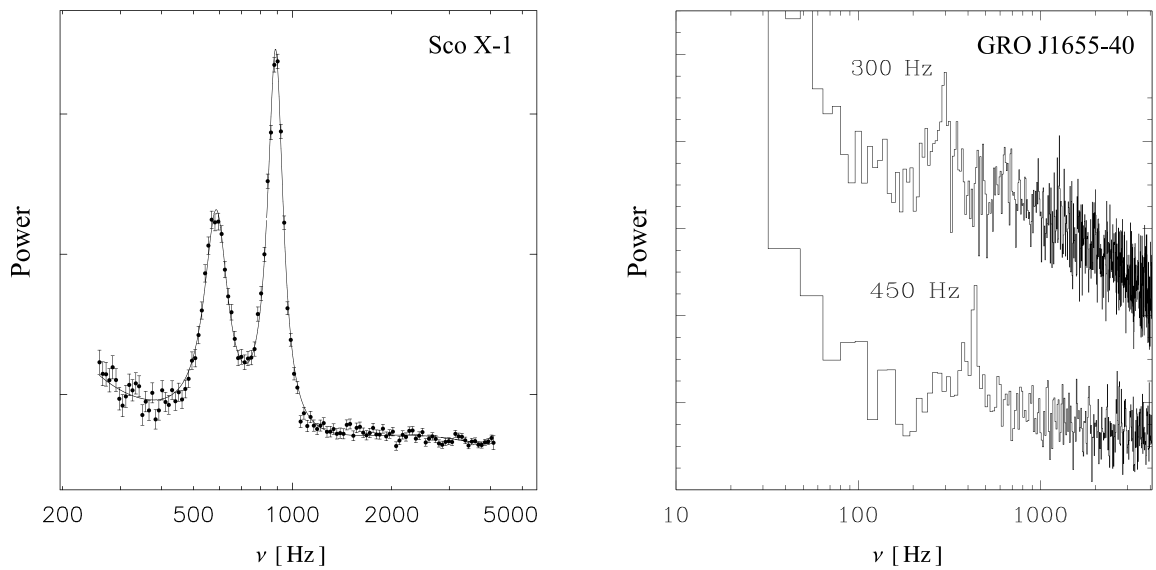



Fig. 1. Twin peak QPOs. Note the 3:2 ratio of the frequencies. *Left*: Typical neutron star frequencies, $\sim 600\,$Hz and $900\,$Hz (from van der Klis et al. 1997). *Right*: Frequencies in a black hole (from Strohmayer 2001).

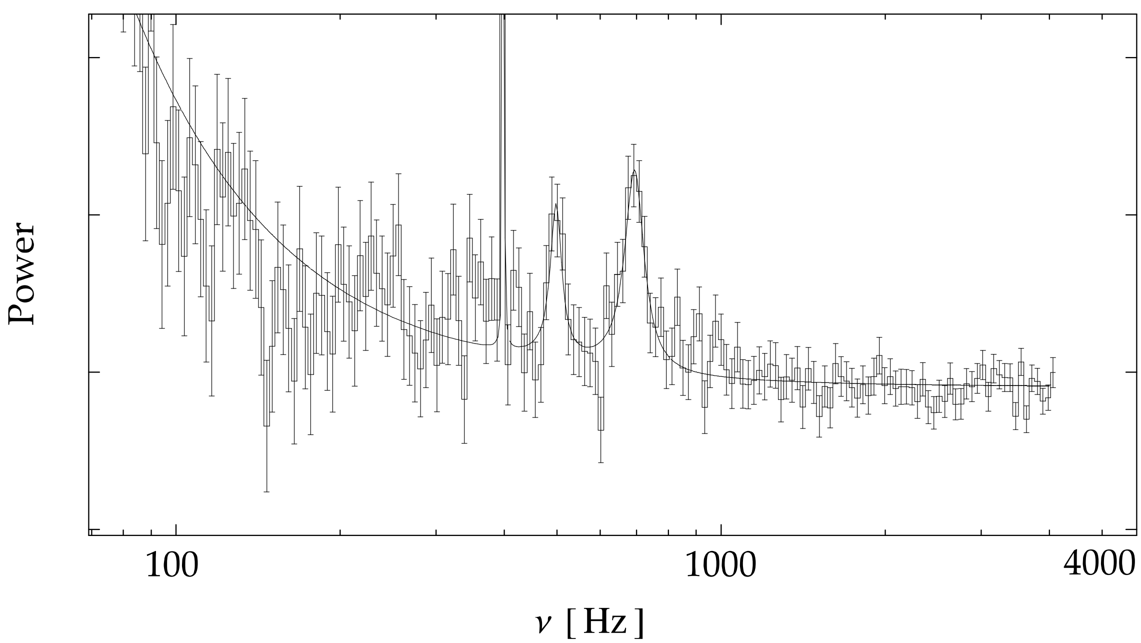



Fig. 2. Two HF QPOs in a 401 Hz accreting millisecond pulsar (from Wijnands et al. 2003). The QPO frequencies, about 700 and 500 Hz, differ by one-half the spin frequency.

oscillators, are listed in Section 2, as are the results obtained. The results are not as certain as the "intrinsic mathematical" ones, and it is likely that detailed interpretations in terms of accretion physics may change in the future.

### 1.1. *What is fully explained by pure mathematics of resonance*

At night, all cats are the same — black. In Nature, all weakly coupled non-linear oscillators in resonance are the same — in their mathematical description. Without knowing the internal structure of pendula, bridges, airplane wings, or accretion disks, one may explain many oscillatory properties of these very different systems, using a *purely mathematical* theory of small non-linear oscillations. Our QPO resonance model is based exactly on this principle. It should be therefore considered natural (rather than surprising) that the resonance model *fully* explains several observational properties of QPOs (including a few considered most fundamental by the observers) directly from the mathematical theory of small non-linear oscillations, *without* reference to the detailed physics of accretion flows. The first three of these properties, which are particularly relevant to neutron star kHz QPOs, find their basis in the mathematical formalism given in Section 5.

[1] **The frequency-frequency correlation (the Bursa line)**. For a given neutron star source, the upper and lower frequencies are linearly correlated along a "Bursa line,"

$$\nu_U = A\,\nu_L + B, \tag{1}$$

with $A \neq 3/2$. Data points occupy a finite sector of the Bursa line (Figure 3). We will refer to the intersection of the Bursa line with the reference line $\nu_U = (3/2)\,\nu_L$ as the "resonance point," because, within the resonance model, at this point the frequencies are equal to the eigen-frequencies of the resonant modes (Abramowicz et al. 2003b; Rebusco 2004; Horák 2004a).

[2] **The slope-intercept anti-correlation**. For a sample of several neutron star sources, the coefficients $A\,, B$ of the Bursa lines (eq. 1) corresponding to individual sources in the sample are anticorrelated,

$$A = 3/2 - B/\,\nu_1, \tag{2}$$

with$\nu_1 = 600\,\mathrm{Hz} \pm \Delta\nu$ where $\Delta\nu \ll 600\,\mathrm{Hz}$ is a small scatter (Figure 4: Right; Abramowicz et al. 2005, 2006a).

[3] **The rational frequency ratio**. As first noticed by Abramowicz & Kluźniak (2001), and confirmed by several other authors (Miller et al. 2001; Remillard et al. 2002; see Török et al. 2005 for most recent references), for black hole sources the frequencies $\nu_U$ and $\nu_L$ are commensurable,[8] and their ratio fixed at $m/n = 3/2$. For neutron star sources the ratio varies in time, but its statistical distribution peaks up, within a few percent, at the 3/2 value (Figure 4: Left; Abramowicz et al. 2003a; Belloni, Méndez, & Homan 2005). Recent evidence suggests that the black-hole HF QPO frequencies may also vary: Homan (2006) detects on occasion a single 260 Hz QPO in GRO J1655-40, which he identifies as an orphaned lower QPO of the usual pair (300 Hz, 450 Hz).

[4] **A jump in rms amplitude difference**. The difference between the rms amplitudes of the upper and lower QPO peaks, $\Delta\,\mathrm{rms}$, is positive on one side of the "resonance point" and negative on the other (Figure 5; Török, Barret, & Horák 2006; Horák et al. 2006b).

### 1.2. *What is also explained by the resonance model*

[5] **The fast and slow rotators: QPOs and neutron star spin**. In outburst, transient accreting LMXBs often display coherent pulsations, whose frequency, $\nu$, is interpreted as the spin rate of the accreting neutron star (Wijnands & van der Klis 1998). When they display twin HF QPOs, the frequency difference of the QPOs is equal to the spin for the slow rotators ($\nu \leq 400\,\mathrm{Hz}$), while for the fast rotators ($\nu \geq 400\,\mathrm{Hz}$) it is equal to to one-half the spin difference (Figure 2); a similar effect occurs in persistent LMXBs, if the burst frequency is interpreted as the spin frequency (Wijnands et al. 2003; Linares et al. 2005). Within the resonance model this is interpreted as evidence for non-linear forcing of the oscillations by the magnetic dipole of the spinning neutron star—when the forcing frequency exceeds the eigen-frequency difference of the two resonant modes, the strongest resonant response occurs when the eigen-frequencies are separated by one-half the forcing frequency (Kluźniak et al. 2004; Lee, Abramowicz, & Kluźniak 2004).

[6] **Sub-harmonics**. Sub-harmonics (i.e., frequencies $\nu_0/n$, with $n = 2, 3, ...,$ where $\nu_0$ is the fundamental) are a hallmark of non-linear resonance, and in addition to the sub-harmonic relationship between the frequencies of fast rotators (Figure 2), there is evidence of subharmonics in black-hole HF QPOs (Remillard et al. 2002).

[7] **Oscillation of QPO frequency**. Yu, van der Klis, & Yonker (2001) report that the frequency of

[8] The resonance model predicts that the ratio should be $m/n$ with $m, n$ being small integers. Several values are possible, and several values may have been observed (e.g. 3/2, 5/3, 2/1). In strong gravity the $m/n = 3/2$ resonance is expected to be the most prominent one (Kluźniak & Abramowicz 2002; Section 3, below), and this ratio is indeed observed far more often than the other ones.

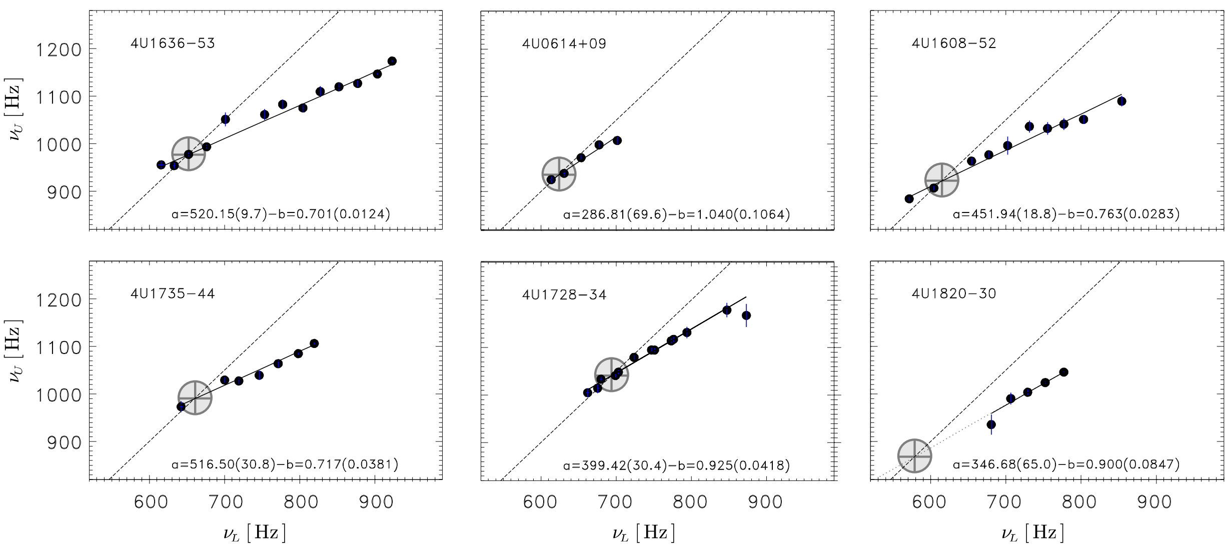



Fig. 3. For individual neutron star sources, the upper $\nu_U$ and lower $\nu_L$ twin peak QPO frequencies are correlated along the straight "Bursa lines", $\nu_U = A\nu_L + B$, with $A \neq 1.5$. Also shown is the line $\nu_U = (3/2)\nu_L$. This figure is from Abramowicz et al. 2006a.

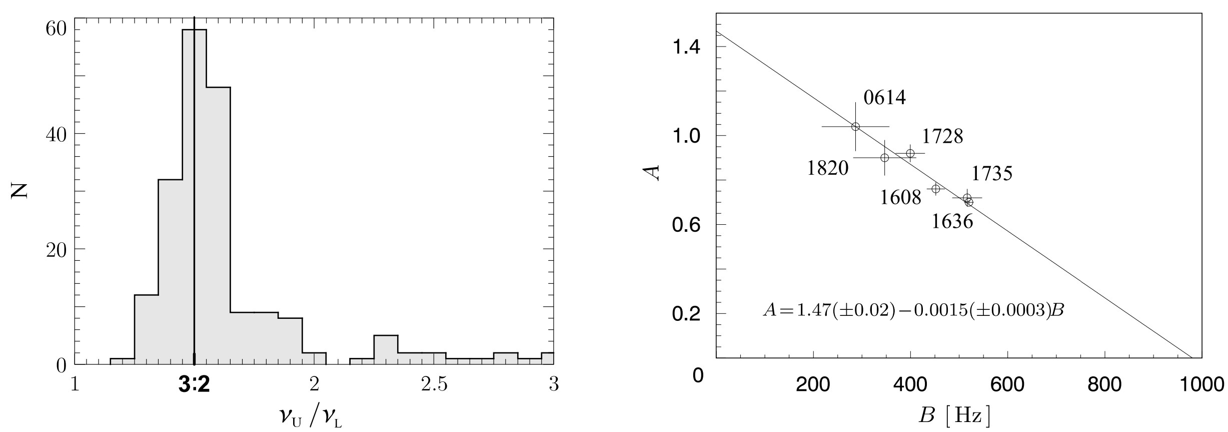



Fig. 4. *Left:* The histogram of frequency ratios observed in neutron stars has a prominent peak at the 3:2 value (after Abramowicz et al. 2003a). *Right:* The data point directly to a 3/2 ratio: the coefficients $A$ and $B$ of the individual Bursa lines, $\nu_U = A\nu_L + B$ are anti correlated, and obey $A = 3/2 - \gamma B$, with $\gamma \approx 1/(600\,\mathrm{Hz})$ (from Abramowicz et al. 2006a).

the upper kHz QPO, as well as the relative amplitudes of both twin QPOs, vary periodically together with the count rate. Such behavior is expected in a conservative system of two coupled oscillators in resonance (Horák et al. 2004b).

[**8**] $1/M$ **scaling**. Another very important aspect of HF QPOs is the inverse mass scaling reported by McClintock & Remillard (2006). This is explained by the mathematics of general relativity, where all frequencies scale inversely with the mass of the black hole, or neutron star. All relativistic models of QPOs have this property, even if they do not involve oscillations of the accretion disk (e.g., Kluźniak, Michelson, & Wagoner 1990; Stella & Vietri 1999). Within our resonance model, the resonance occurs between two modes of an accretion disk or a torus computed in general relativity (e.g., Wagoner 1999; Abramowicz et al. 2006b), so the $1/M$ scaling is a built-in feature of the model.

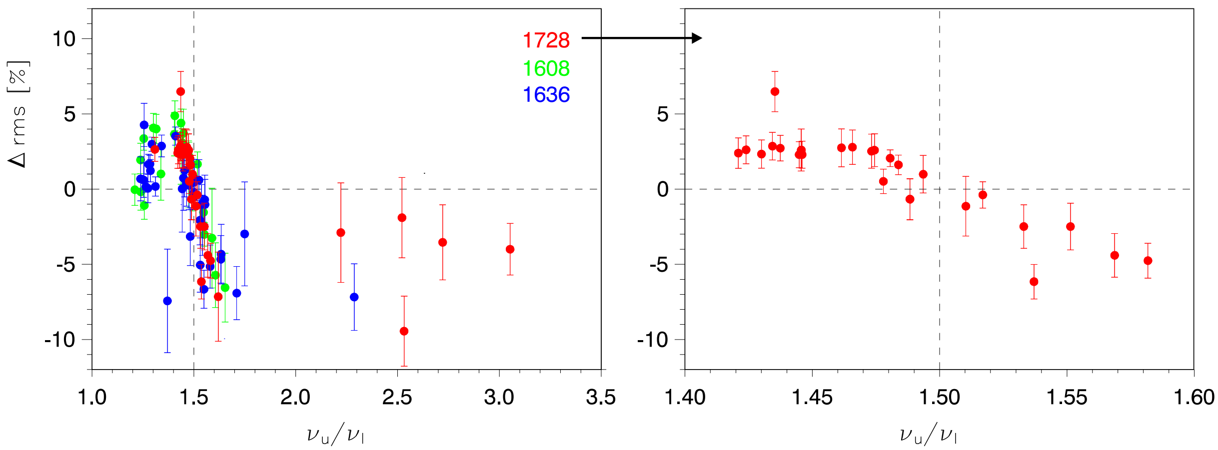



Fig. 5. The difference in rms amplitudes of the twin kHz QPOs undergoes a sign change as the frequency ratio crosses the 3/2 resonance (from Török, Barret, & Horák, 2006). The data analysis technique has been described in Barret et al. 2005b,2006.



### 1.3. *What may yet be explained by the resonance model*

Examples of those intrinsic QPO properties that have not yet been explained by pure mathematics of the resonance model, but most likely will be, are the behavior of the quality factor $Q$ of the neutron star kHz QPOs (Barret et al. 2005a,b), and the correlation of kHz frequencies with those of lower-frequency features in the power density spectrum (Psaltis, Belloni, & van der Klis 1999; Mauche 2002; Warner, Woudt, & Pretorius 2003).

## 2. WHAT CANNOT BE EXPLAINED BY PURE MATHEMATICS

An example of an important, and still unexplained, property of QPOs that depends on factors external to the mathematics of coupled oscillators is the remarkable connection of QPOs with the spectral states of black hole and neutron star sources (Homan et al. 2001; Remillard et al. 2002).

Most likely, it depends on detailed physics that governs excitation, damping and modulation. A full explanation will involve radiative transfer calculations that will only be possible to carry out when the full structure of the accretion disk is known in detail.

Some other (fundamental) issues and questions that (most likely) cannot be explained by pure mathematics of the theory of small oscillations alone, and must be studied within a detailed physical model of accretion flow, are listed below.

*[a] How is the X-ray flux modulated by the oscillations of the accretion disk that give rise to QPOs?* A full answer to this question is only possible after another question is answered:

*[b] What two modes are in resonance?* An often discussed possibility (first suggested by Kluźniak & Abramowicz 2002) is that these are epicyclic modes of slender tori, and this model is reviewed in the companion article (Kluźniak et al. 2007). See Zanotti, Rezzola, & Font 2003; Blaes et al. 2006; Abramowicz et al. 2006b for related models. The corresponding modulation mechanism in black holes would be light bending (Bursa et al. 2004; Schnittman & Rezzolla 2006). However, this is not firmly established and there are other possibilities. Kato (2004a,b,2005) discusses non-linear coupling of g-modes of a standard thin disk, with the resonance mediated by a warp postulated to be present in the disk.

*[c] How are the modes excited and coupled? What is the energy source that feeds the resonance?* It has been suggested (Kluźniak et al. 2004; Lee et al. 2004) that the excitation in the neutron star case could be effected by the neutron star spin (e.g., due to a magnetic coupling), and in the black hole case by the influence of turbulence (Abramowicz et al. 2005; Brandenburg 2005; Vio et al. 2006).

## 3. THE FREQUENCIES OF STRONG GRAVITY: KEPLERIAN AND EPICYCLIC

Epicyclic frequencies of free *particles* enter our discussion of *fluid* oscillations in accretion flows, because some important modes of fluid oscillations in

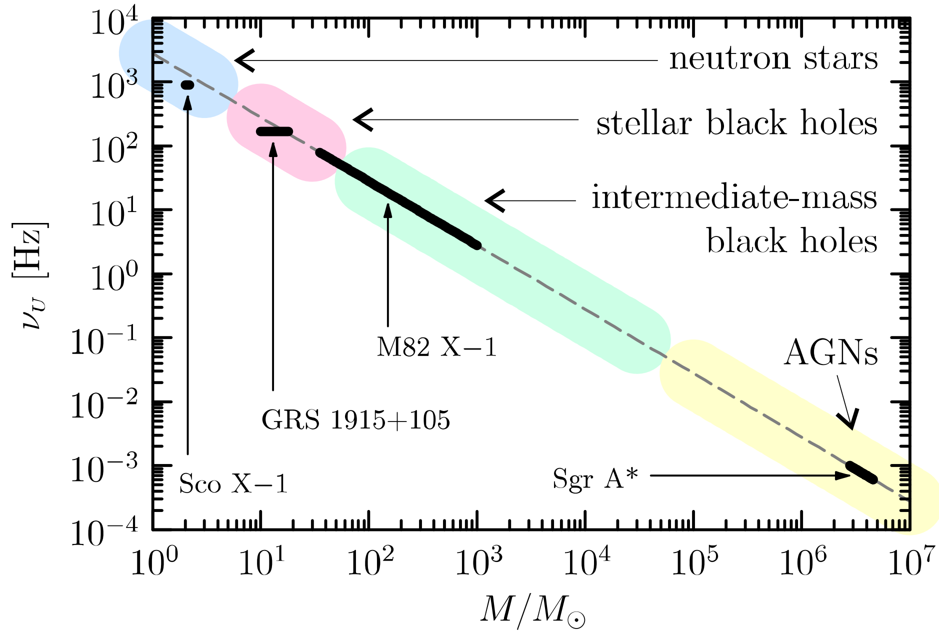



Fig. 6. QPO frequencies expected on $1/M$ scaling for different systems (after Bursa 2006).

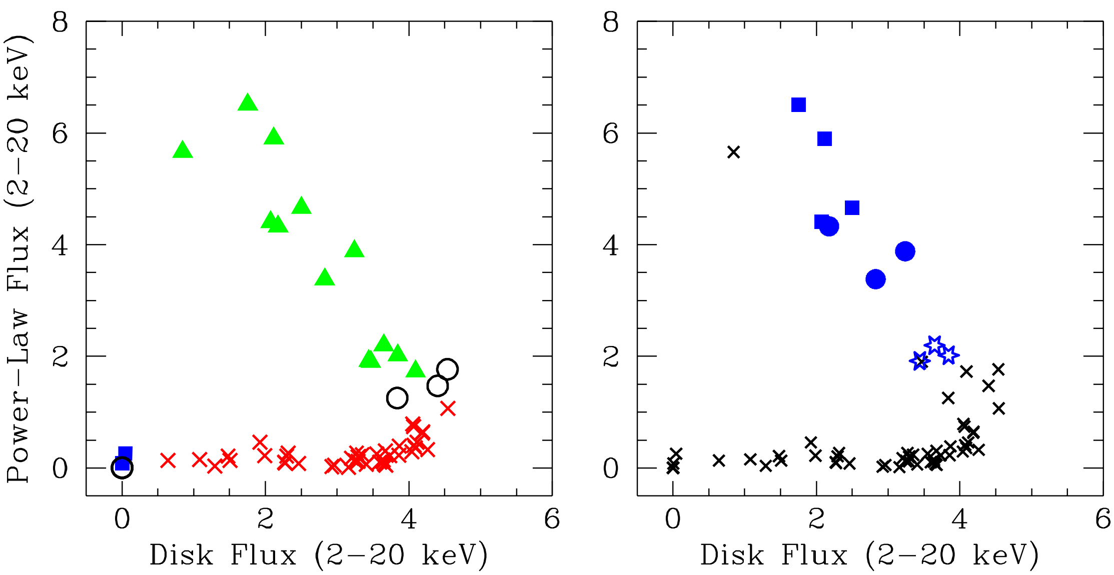
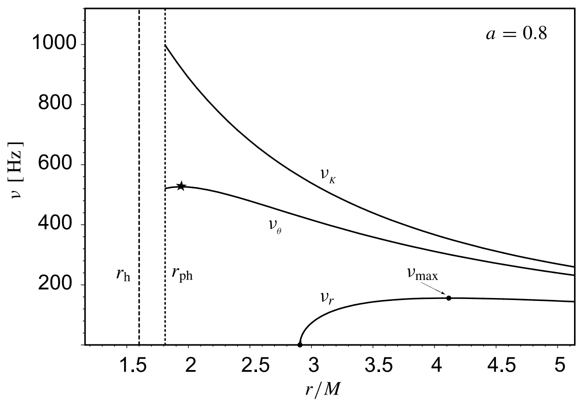



Fig. 7. X-ray states and HFQPOs during the 1996-1997 outburst of GRO J1655–40. The left panel shows the energy diagram, where flux from the accretion disk is plotted versus flux from the power-law component. Here, the symbol type denotes the X-ray state: thermal (red "x"), hard (blue square), steep power-law (green triangle), and any type of intermediate state (yellow circle). The right panel shows the same data points, while the symbol choice denotes HFQPO detections: 300 Hz (blue squares), 450 Hz (blue star), both HFQPOs (blue circle), and no HFQPO (black "x"). The HFQPO detections are clearly linked to the SPL state, and the HFQPO frequency is clearly correlated with power-law luminosity. From Remillard et al. 2002.

accretion flows (that may be directly relevant for QPOs), have their eigen-frequencies nearly equal to the epicyclic frequencies in nearly Keplerian motion of free particles (Kluźniak & Abramowicz 2002; Abramowicz et al. 2003b, 2006b; Blaes et al. 2006). Small perturbations of these flows are given in the



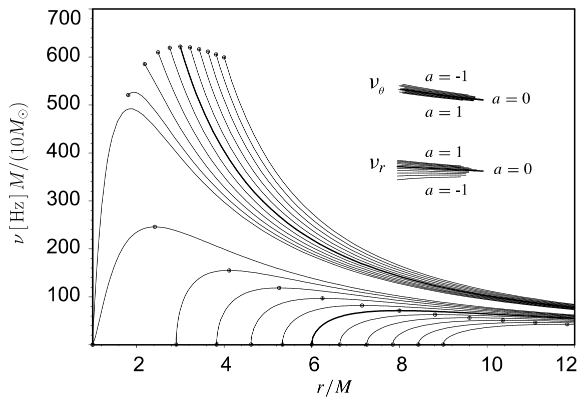




Fig. 8. *Left:* The orbital and epicyclic frequencies for a $10M_\odot$ black hole with angular momentum $0.8M^2G/c$. Also shown are the locations of the horizon and the photon orbit. *Right:* The epicyclic frequencies in Kerr metric.



lowest order by two uncoupled harmonic, "epicyclic" oscillations,

$$\delta\ddot{r} + (\omega_r)^2\delta r = 0, \quad (3)$$
$$\delta\ddot{\theta} + (\omega_\theta)^2\delta\theta = 0. \quad (4)$$

In weak Newtonian gravity with the $-GM/r$ potential the three characteristic orbital frequencies, Keplerian $\Omega_K$, radial epicyclic $\omega_r$ and vertical epicyclic $\omega_\theta$, are all equal. In strong gravity, $\Omega_K \geq \omega_\theta > \omega_r$ for pro-grade orbits, and $\omega_\theta \geq \Omega_K > \omega_r$ for retrograde orbits (Figure 8).

### 3.1. *The strong gravity Mathieu equation*

Even for a nearly-Keplerian flow pressure and other stresses may provide a non-linear coupling, and the simplest one is $\delta r\,\delta\theta$ in eq. (4). In this case, the new version of eq. (4) may be combined with the solution of eq. (3) into the Mathieu form,

$$\delta\ddot{\theta} + (\omega_\theta)^2[1 + A\cos(\omega_r\, t)]\delta\theta = 0. \quad (5)$$

From the theory of Mathieu equation one knows that a *parametric resonance* occurs when $n\,\omega_r = 2\,\omega_\theta$. Because in strong gravity $\omega_r < \omega_\theta$, the smallest value of $n$ consistent with the resonance is $n = 3$, and the resonant ratio is $\omega_\theta/\omega_r = 3/2$. Note that the particular value 3/2 is a direct consequence of the strong gravity. For a more complete discussion, see Kluźniak & Abramowicz 2002, Abramowicz et al. 2003b, Kluźniak 2005.

## 4. CONCLUSIONS

General features of the mathematical theory of small non-linear oscillations should be taken seriously in *any* theoretical study of QPOs. Suppose that a particular observed QPO property, [1]-[7] of Section 1.1, or another one, was theoretically explained in the framework of a very well determined model that employs some very specific, detailed and sophisticated "real physics" based, e.g., on thin disk diskoseismology, MHD accretion flows, or warped disk precession. Does this mean that this particular "real physics" is supported by the QPO observations? Not necessarily. It often happens that the derived property is not a signature of the particular physics that was assumed, but is instead a purely mathematical property of small non-linear, weakly coupled oscillations in resonance. A different model, with a different physics, would recover this property as well.

## 5. APPENDIX: MATHEMATICS OF THE QPO RESONANCE MODEL

Although the simplest $\delta r\,\delta\theta$ coupling may not be realistic, a parametric Mathieu-type resonance very similar to the one discussed in Section 3.1 typically occurs in a wide class of physically possible nearly-Keplerian accretion flow. This was first carefully demonstrated by Rebusco (2004), who realized that the most convenient mathematical tool here should be the multiple scales method. Her work was then generalized by Horák (2004a, 2005), Horák & Karas (2006a) and others. They considered the most general mathematical model of coupled oscillators, and made no mathematical simplifications. These results provide the hard mathematical core for the resonance model.

### 5.1. *Equations*

The resonance model studies modes of small amplitude oscillations in nearly-Keplerian accretion flows. Oscillations occur near an "equilibrium" in which fluid elements move on circular orbits. The Lagrangean displacement from the equilibrium has the components,

$$
\begin{aligned}
\delta r(s) &= r(s) - r_0, && (6)\\
\delta\theta(s) &= \theta(s) - \theta_0, && (7)\\
\delta\varphi(s) &= \varphi(s) - (u_0^\varphi + \delta u_0^\varphi)s, && (8)\\
\delta t(s) &= t(s) - (u_0^t + \delta u_0^t)s, && (9)
\end{aligned}
$$

where the index zero denotes a constant quantity in equilibrium. The displacement components obey three independent coupled, non-linear, second order ordinary differential equations, consistent with three coupled, forced and damped, anharmonic oscillators,

$$
\begin{aligned}
\delta\ddot{r} + (\omega_r^0)^2\delta r &= \mathcal{X}_r(\delta x^k, \delta\dot{x}^k) + \mathcal{F}_r(x^k, u^k, s), && (10)\\
\delta\ddot{\theta} + (\omega_\theta^0)^2\delta\theta &= \mathcal{X}_\theta(\delta x^k, \delta\dot{x}^k) + \mathcal{F}_\theta(x^k, u^k, s), && (11)\\
\delta\ddot{\varphi} + (\omega_\varphi^0)^2\delta\varphi &= \mathcal{X}_\varphi(\delta x^k, \delta\dot{x}^k) + \mathcal{F}_\varphi(x^k, u^k, s) && (12)
\end{aligned}
$$

For symmetry, it is sometime convenient to consider the fourth equation,

$$\delta\ddot{t} + (\omega_t^0)^2\delta t = \mathcal{X}_t(\delta x^k, \delta\dot{x}^k) + \mathcal{F}_t(x^k, u^k, s). \quad (13)$$

Equations (10)–(13) are not independent, however, because of $u^i u_i = 1$. In (10)-(13) a dot denotes differentiation with respect to the proper time $s$.

### 5.2. *Expansion, multiple scales, solutions*

Let us study nonlinear oscillations of the system having two degrees of freedom, i.e., the coordinate perturbations $\delta r$ and $\delta\theta$. The oscillations are described by two coupled differential equations of the very general form

$$
\begin{aligned}
\ddot{\delta r} + \omega_r^2\,\delta r &= \omega_r^2\, f_r(\delta r, \delta\theta, \dot{\delta r}, \dot{\delta\theta}), && (14)\\
\ddot{\delta\theta} + \omega_\theta^2\,\delta\theta &= \omega_\theta^2\, f_\theta(\delta r, \delta\theta, \dot{\delta r}, \dot{\delta\theta}). && (15)
\end{aligned}
$$

Suppose that the functions $f_r$ and $f_\theta$ are nonlinear, i.e., their Taylor expansions start in the second order. Another assumption is that these functions are invariant under reflection of time (i.e., the Taylor expansion does not contain odd powers of time derivatives of $\delta r$ and $\delta\theta$). As we see later, this assumption is related to the conservation of energy in the system. Many authors studied such systems with a particular form of functions $f$ and $g$ (Nayfeh & Mook 1979), however, in this paper we keep discussion fully general.

We seek the solutions of the governing equations in the form of the multiple-scales expansions (Nayfeh & Mook 1979)

$$\delta r(t,\epsilon) = \sum_{n=1}^{4} \epsilon^n r_n(T_\mu), \quad \delta\theta(t,\epsilon) = \sum_{n=1}^{4} \epsilon^n \theta_n(T_\mu), \quad (16)$$

where several time scales $T_\mu$ are introduced instead of the physical time $t$,

$$T_\mu \equiv \epsilon^\mu t, \quad \mu = 0, 1, 2, 3. \quad (17)$$

The time scales are treated as independent. It follows that instead of the single time derivative we have an expansion of partial derivatives with respect to the $T_\mu$

$$
\begin{aligned}
\frac{d}{dt} &= 0 + \epsilon 1 + \epsilon^2 2 + \epsilon^3 3 + \mathcal{O}(\epsilon^4), && (18)\\
\frac{d^2}{dt^2} &= 0^2 + 2\epsilon 01 + \epsilon^2(1^2 + 202) + \\
&\quad 2\epsilon^3(03 + 12) + \mathcal{O}(\epsilon^4), && (19)
\end{aligned}
$$

where $\mu = \partial/\partial T_\mu$.

We expand the nonlinear functions $f_r$ and $f_\theta$ into the Taylor series and then we substitute the expansions (16), (18) and (19). Finally, we compare the coefficients of the same powers of $\epsilon$ on both sides in the resulting couple of equations. This way we get a set of *linear* second-order differential equations that can be solved successively – the lower-order terms of the expansion (16) appear as forcing terms on the right-hand sides in the equations for the higher order approximations.

In the first order we obtain equations corresponding to the linear approximation

$$(0^2 + \omega_r^2)r_1 = 0, \quad (0^2 + \omega_\theta^2)\theta_1 = 0. \quad (20)$$

with the solutions

$$
\begin{aligned}
r_1 &= A_r(T_1, T_2, T_3)e^{i\omega_r T_0} + \text{cc}, && (21)\\
\theta_1 &= A_\theta(T_1, T_2, T_3)e^{i\omega_\theta T_0} + \text{cc}. && (22)
\end{aligned}
$$

The complex amplitudes $\widehat{A}_r$ and $A_\theta$ generally depend on the higher time-scales.

### 5.3. *Resonances*

The solutions (21) and (21) substituted into the quadratic terms in the right-hand side of the second-order differential equations produces terms that oscillates with frequencies $2\omega_r$, $2\omega_\theta$ and $\omega_\theta \pm \omega_r$. When the frequency ratio $\omega_r/\omega_\theta$ is far from $1:2$ and $2:1$ the solutions $r_2$ and $\theta_2$ describe higher harmonics to the linear-order oscillations $r_1$ and $\theta_1$. Hence, the presence of higher harmonics in the power-spectra is a general signature of nonlinear oscillations. Their frequencies and relative strengths with respect to the main oscillations could provide us usefull informations about nonlinearities in the system.

In addition, the right hand sides of the second order equations contain terms proportional to $e^{i\omega_r T_0}$ and $e^{i\omega_\theta T_0}$ that oscillate with the same frequency as the eigen-frequency of the oscillators. These terms produce secular grow of the amplitudes of the second-order approximations $r_2$ and $\theta_2$ and cause nonuniform expansions (16). Eliminating them we get the *solvability conditions* for the complex amplitudes $A_r(T_1, T_2, T_3)$ and $A_\theta(T_1, T_2, T_3)$ that give us the evolution of the system on longer time-scales (Nayfeh & Mook 1979).

When the eigen-frequencies are in 1:2 or 2:1 ratio we observe qualitatively different behavior related to the *autoparametric resonance*. In that case the right hand sides contain additional secular terms and the solvability conditions take different form. Different resonances occur in different orders of approximation. The possible resonances in the third order are 1:3, 1:1 and 3:1 and 1:4, 3:2, 2:3 and 4:1 in the fourth order.[9] However, if the governing equations remain unchanged under the transformation $\delta\theta \to -\delta\theta$ (i.e., the system is reflection symmetric) the only autoparametric resonances that exists in the system are 1:2, 1:1, 1:4 and 3:2 (Rebusco 2004).

### 5.4. *Conservative systems*

In non-linear systems, oscillation frequencies depend on the amplitudes,

$$\omega^{(a)} = \omega_0^{(a)} + f^{(a)}(\xi). \tag{23}$$

Here, $\xi$ is a parameter. When the external influence is small, $\mathcal{F}^i \le \mathcal{O}(\delta x^3)$, the energy of the oscillations is conserved (Horák 2004a),

$$E = [A^{(1)}]^2 + k[A^{(2)}]^2 = \mathrm{const} + \mathcal{O}(\delta x^3). \tag{24}$$

This implies that the amplitudes are correlated, and can be expressed in a parametric form,

$$[A^{(1)}(\xi)]^2 = \frac{(2\xi + \xi_0)\,E}{2(1+\xi_0)}, [A^{(2)}(\xi)]^2 = \frac{(2 - 2\xi + \xi_0)\,E}{2k(1+\xi_0)}, \tag{25}$$

with a particular choice of the constant $\xi_0$ that assures $f^{(a)}(\xi) = 0$ for $\xi = 0$. Therefore,

$$\omega^{(a)} = \omega_0^{(a)} + [f^{(a)}]'\,\xi + \dots, \tag{26}$$

[9] The ratio $n : m$ refers to the eigen-frequency ratio $\omega_\theta : \omega_r$.

where the prime denotes differentiation with respect to $\xi$ at $\xi = 0$. From this it follows directly that the frequencies are linearly correlated (Rebusco 2004),

$$\omega^{(1)} = A\,\omega^{(2)} + B. \tag{27}$$

where the slope $A$ and the intercept B are

$$A = \frac{[f^{(1)}]'}{[f^{(2)}]'}, \quad B = -A\,\omega_0^{(2)} + \omega_0^{(1)}. \tag{28}$$

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## REFERENCES

Abramowicz, M. A., Barret, D., Bursa, M., Horák, J., Kluźniak, W., Rebusco, P., & Török, G. 2005, Astron. Nach., 326, 864
Abramowicz, M. A., Barret, D., Bursa, M., Horák, J., Kluźniak, W., Olive, J.-F., Rebusco, P., & Török G. 2006a, submitted
Abramowicz, M. A., Blaes, O. M., Horák, J., Kluźniak, W., & Rebusco, P. 2006b, CQGra, 23, 1689
Abramowicz, M. A., Bulik, T., Bursa, M., & Kluźniak, W. 2003a, A&A, 404, L21
Abramowicz, M. A., Karas, V., Kluźniak, W., Lee, W. H., & Rebusco, P. 2003b, PASJ, 55, 467
Abramowicz, M. A., & Kluźniak, W. 2001, A&A, 374, L19
Belloni, T., Méndez, M., & Homan, J. 2005, A&A, 437, 209
Blaes, O. M., et al. 2006, in preparation
Barret, D., Kluźniak, W., Olive, J. F., Paltani, S., & Skinner, G. K. 2005a, MNRAS, 357, 1288
Barret, D., et al. 2005b, Astr. Nach., 326, 808
Barret, D., et al. 2006, MNRAS, in press
Brandenburg, A. 2005, Astr. Nach., 326, 787
Bursa, M. 2006, PhD Thesis, Charles University, Prague
Bursa, M., Abramowicz, M. A., Karas, V., & Kluźniak, W. 2004, ApJ, 617, L45
Homan, J. 2006, personal comm.
Homan, J., Wijnands, R., van der Klis, M., Belloni, T., van Paradijs, J., Klein-Wolt, M., Fender, R., & Méndez, M. 2001, ApJS, 132, 377
Horák, J. 2004a, in Processes in the Vicinity of Black Holes and Neutron Stars, 4, ed. S. Hledík, & Z. Stuchlík (Opava: Silesian University), 91
Horák, J., Abramowicz, M. A., Karas, V., & Kluźniak, W. 2004b, PASJ, 56, 819
Horák, J. 2005, Astr. Nach. 326, 824
Horák, J., & Karas, V. 2006a, A&A, 451, 377
Horák, J., Rebusco, P., Abramowicz, M. A., & Török, G. 2006b, in preparation
Kato, S. 2004a, PASJ, 56, L25

__________. 2004b, PASJ, 56, 905
__________. 2005, PASJ, 57, 699
Kluźniak, W. 2005, Astr. Nach., 326, 820
Kluźniak, W., & Abramowicz. M. 2000, Phys. Rev. Lett., submitted (astro-ph/0105057)
__________. 2001, Acta Phys. Polonica B, 32, 3605
__________. 2002, astro-ph/0203314
Kluźniak, W., Abramowicz, M. A., Bursa, M., & Török G. 2007, RevMexAA (SC) 27, 19
Kluźniak, W., Abramowicz, M. A., Kato, S., Lee, W. H., & Stergioulas, N. 2004, ApJ, 603, L89
Kluźniak, W., Michelson, P., & Wagoner, R. V. 1990, ApJ, 358, 538
Lee, W. H., Abramowicz, M. A., & Kluźniak, W. 2004, ApJ, 603, L93
Linares, M., van der Klis, M., Altamirano, D., & Markwardt, C. B. 2005, ApJ, 634, 1250
Mauche, C. W. 2002, ApJ, 580, 423
McClintock, J. E., & Remillard, R. A. 2006, in Compact Stellar X-ray Sources, ed. W.H.G. Lewin & M. van der Klis (Cambridge: Cambridge Univ. Press)
Miller, J. M., et al. 2001, ApJ, 563, 928
Nayfeh, A. H., & Mook, D. T. 1979, Nonlinear Oscillations (New York: Wiley-Interscience)
Psaltis, D., Belloni, T., & van der Klis, M. 1999, ApJ, 520, 262
Rebusco, P. 2004, PASJ, 56, 533
Remillard, R. A., Muno, M. P., McClintock, J. E., & Orosz, J. A. 2002, ApJ, 580, 1030
Schnittman, J. D., & Rezzolla, L. 2006, ApJ, 637, L113
Stella, L., & Vietri, M. 1999, Phys. Rev. Lett., 82, 17
Strohmayer, T. 2001, ApJ, 552, L49
Török, G., Abramowicz, M. A., Kluźniak, W., & Stuchlík, Z. 2005, A&A, 436, 1
Török, G., Barret, D., & Horák, J. 2006, in preparation
van der Klis, M., Wijnands, R. A. D., Horne, K. K., & Chen, W. 1997, ApJ, 481, 97
Vio, R., Rebusco, P., Andreani, P., Madsen, H., & Overgaard, R. 2006, A&A, 452, 383
Wagoner, R. W. 1999, Phys. Rep., 311, 259
Warner, B., Woudt, P., & Pretorius, M. L. 2003, MNRAS, 344, 1193
Wijnands, R., et al. 2003, Nature, 424, 44
Wijnands, R., & van der Klis, M. 1998, Nature, 394, 344
Yu, W., van der Klis, M., & Jonker, P. G. 2001, ApJ, 559, L29
Zanotti, O., Rezzolla, L., & Font, J. A. 2003, MNRAS, 341, 832



Marek Abramowicz: Department of Physics, Göteborg University, S-412 96 Göteborg, Sweden (Marek@fy.chalmers.se).
Michal Bursa and Jiří Horák: Astronomical Institute, Academy of Sciences, Boční II 1401, CZ-141 31 Praha 4, Czech Republic (bursa@astro.cas.cz, jhorak@math.uni-koeln.de).
Włodek Kluźniak: Copernicus Astronomical Centre, ul. Bartycka 18, Warszawa, 00-716, Poland (wlodek@camk.edu.pl).
Paola Rebusco: Max-Planck-Institute für Astrophysik, Karl-Schwarzchild-Str. 1, D-85741 Garching, Germany (prebusco@mpa-garching.mpg.de).
Gabriel Török: Institute of Physics, Silesian University in Opava, Bezručovo nám. 13, CZ-746 01 Opava, Czech Republic (terek@volny.cz).