

PAPAPETROU'S EQUATIONS OF MOTION FOR AN EXTENDED TEST BODY

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RESUMEN

A partir de las ecuaciones de movimiento de Dixon para cuerpos extendidos, calculamos las ecuaciones de movimiento de Papapetrou para un cuerpo extendido de prueba en un espacio-tiempo de Schwarzschild, incorporando los términos relacionados con la fuerza y el torque, los cuales involucran momentos multipolares de orden superior al dipolar. Con el fin de completar el sistema, introducimos la condición suplementaria de espín de Corinaldesi-Papapetrou obteniendo como resultado las ecuaciones de movimiento en el sistema en reposo del campo de Schwarzschild.

ABSTRACT

We use Dixon's general equations of motion for extended bodies to compute the Papapetrou's equations for an extended test body in Schwarzschild space-time. We incorporate the force and torque terms which involve multipolar moments. The Corinaldesi-Papapetrou spin supplementary condition is introduced to obtain the equations of motion in the rest frame of the Schwarzschild field.

Key Words: celestial mechanics — gravitation — methods: analytical

1. INTRODUCTION

The general equations of motion for an extended body in a given gravitational background were obtained by Dixon (Dixon 1970, 1974) in multipole approximation of the body structure for any order. The method involves the definition of a world-tube enclosing the entire body, an appropriate foliation of the spacetime and a convenient worldline representing the center of mass (CM), around which the multipole expansion are performed. The set of equations for extended bodies is

$$F^\nu \equiv \frac{\delta p^\nu}{\delta s} - \frac{1}{2} S^{\kappa\lambda} v^\mu R_{\kappa\lambda\mu}{}^\nu, \quad (1)$$

$$L^{\kappa\lambda} \equiv \frac{\delta S^{\kappa\lambda}}{\delta s} - 2p^{[\kappa} v^{\lambda]}, \quad (2)$$

where $p^\nu = Mu^\nu$ is the total four-momentum, $S^{\kappa\lambda}$ is the spin tensor, F^ν and $L^{\kappa\lambda}$ are the force and torque linked with the structure of the body beyond the quadrupole terms.

Even before Dixon's work, Papapetrou (Papapetrou 1951; Corinaldesi & Papapetrou 1951) derived the equations of motion of spinning test particles, which are the starting point for the analysis of the precession of gyroscopes. We consider these equations as a particular case of the equations (1-2),

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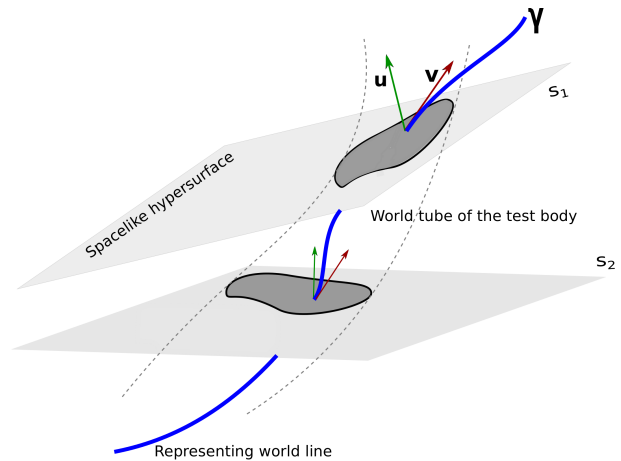


Fig. 1. The timelike world line enclosed by a world tube. The spacelike hypersurfaces are orthogonal to $u^\mu = (v^0, \mathbf{u})$ and $v^\mu = (v^0, \mathbf{v})$ is tangent to the world line.

and compute the equations of motion for an extended test body immersed in Schwarzschild spacetime. We impose the Corinaldesi-Papapetrou spin supplementary condition which specifies the line γ representing the motion of the center of mass.

2. NOTATION AND CONVENTIONS

The Schwarzschild metric in Cartesian coordinates (t, x^i) is written as

$$g_{00} = -e^\mu, \quad g_{ij} = \delta_{ij} - \frac{(1 - e^{-\mu})}{r^2} x^i x^j, \quad (3)$$

with $e^\mu = -(1 - 2r_0/r)$, and $r_0 = Gm/c^2$ being the gravitational radius of the central body.

Usually to equations (1-2), supplementary conditions are added which single out the worldline of the CM. We apply the Corinaldesi-Papapetrou condition which holds in the rest-frame of the Schwarzschild field. As a consequence of this choosing the independent components of the spin tensor reduces to three, thus

$$S^k \equiv \frac{1}{2} \epsilon_{ijm} \delta^{km} S^{ij}, \quad (4)$$

3. EQUATIONS OF MOTION

Let λ^α to be a four-vector defined by

$$\begin{aligned} \lambda^0 &= c \frac{2r_0}{r^3} e^{-\mu} r v \dot{t}, \\ \boldsymbol{\lambda} &= \frac{r_0}{r^3} \left[c^2 e^\mu \dot{t}^2 + 2|\mathbf{v}|^2 - \frac{1}{r^2} (2 + e^{-\mu}) r v^2 \right] \mathbf{r}, \end{aligned} \quad (5)$$

and let the torque terms, related to high multipolar structure of the test body,

$$\boldsymbol{\tau} = \frac{1}{2} \epsilon_{ijm} \left(\frac{1}{2} L^{ij} + \frac{1}{v^0} L^{0[iv^j]} \right) \delta^{km}, \quad (6)$$

$$\boldsymbol{\zeta}^i = \frac{\delta}{\delta s} \left(\frac{1}{v^0} L^{0i} \right). \quad (7)$$

Then, the non-geodesic equations of motion can be expressed as

CM equation

$$\frac{d}{ds} (M_* v^0) + M_* \lambda^0 - F^0 = 0, \quad (8)$$

$$\begin{aligned} &\frac{d}{ds} (M_* \mathbf{v}) + M_* \boldsymbol{\lambda} + \frac{3r_0}{r^5} [(\mathbf{S} \cdot \mathbf{r})(\mathbf{r} \times \mathbf{v}) \\ &+ e^{-\mu} (\mathbf{r} \cdot \mathbf{v})(\mathbf{r} \times \mathbf{S})] + \mathbf{F} - \frac{r_0}{r^3} e^{-\mu} (\mathbf{r} \times \boldsymbol{\tau}) - \boldsymbol{\zeta} = 0. \end{aligned} \quad (9)$$

Spin equation

$$\begin{aligned} \dot{\mathbf{S}} - \frac{r_0}{r^3} [2e^{-\mu} (\mathbf{r} \cdot \mathbf{v}) \mathbf{S} + 2(\mathbf{r} \cdot \mathbf{S}) \mathbf{v} - e^{-\mu} (\mathbf{v} \cdot \mathbf{S}) \mathbf{r} \\ - \frac{1}{r^2} (2 + e^{-\mu}) (\mathbf{r} \cdot \mathbf{v})(\mathbf{r} \cdot \mathbf{S}) \mathbf{r}] - \boldsymbol{\tau} = 0. \end{aligned} \quad (10)$$

Where differentiation with respect to the parameter s are denoted with a dot.

3.1. Effective Mass

In the equations (8-10), M_* represents an effective mass associated with the mass of the body plus an energetic component which results from the interaction between the multipolar structure of the body and the spacetime curvature.

$$M_* = M + M_s + M_L, \quad (11)$$

where M is a positive scalar coming from the four-momentum definition. In general it is not constant and its variation depends on the high multipolar structure. Hence, dM/ds vanishes whenever one neglects the force and torque which arise from the multipole moments of the body. M_s assumes the characteristic form of a spin-orbit interaction energy and M_L represents the energy associated with the interaction between the structure of the test body and the gravitational fields. They are written as

$$M_s = \frac{r_0}{Mr^3} e^{-\mu} (\mathbf{r} \times \mathbf{p}) \cdot \mathbf{S}, \quad (12)$$

and

$$M_L = \frac{u_\sigma}{v^0} L^{0\sigma}. \quad (13)$$

4. CONCLUSION

We have examined the Papapetrou's equations of motion for an extended body with arbitrary multipolar structure. The terms related to force and torque in the equations (8-10) are new contributions to the motion of the spinning test body associated with the quadrupole and higher multipolar structure of the body, which depends on its stress-energy tensor and the gravitational fields (Almonacid 2014). In the dipole approximation this equations reduce to the classical Papapetrou's equations. Also, we present an additional contribution to the mass of the body in (13).

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