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THE BAYESIAN CRAMÉR-RAO LOWER BOUND IN PHOTOMETRY

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In photometry, a topic of interest is to estimate the maximum precision that can be achieved by an estimator. In this context we analyse the bounds of precision on a CCD detector array in a Bayesian setting, where we have access to a prior distribution. We use the Bayesian Cramér-Rao (BCR) lower bound to analyse the gain in photometric performance in contrast with the parametric scenario where no prior information is available (or is discarded) for the inference problem.

1. BAYES ESTIMATION

The problem of interest is the inference of the flux of a point source as measured by a detector array in a Bayesian scenario, which considers that the flux $\tilde{F}$ is a random variable as opposed to a fixed although unknown parameter considered in a classical parametric scenario. More precisely, given a set of independent measurements, a basic methodological question is: What is the minimum mean square error (MSE) attainable in the determination of such a random parameter given our data? This question can be solved using fundamental limits employed in statistics, known as the Bayesian Crámer-Rao lower bound (BCR).

2. OBSERVATIONAL SETTING

Consider an intensity profile $\tilde{F}(x, x_c) = \tilde{F}\phi(x - x_c, \sigma)$ where $\phi(x - x_c, \sigma)$ denotes the 1-Dimensional normalized point spread function (PSF). For simplicity, we use the 1-D case, because it is simpler to deal with, and the extension to a 2-D detector is straightforward. In what follows, x_c and σ are assumed to be known and fixed. In practice, $\tilde{F}\phi(x - x_c, \sigma)$ is not observed directly mainly because of three sources of uncertainty that affect all measurements. The first is an additive background $\tilde{B}$ noise which captures the photon emission of the diffuse sky and the noise of the instrument (read-out noise and dark current). The second is the spatial quantization process associated with the pixel resolution (we will assume that the source has a Gaussian-shape profile). The third is an intrinsic uncertainty between the nominal object brightness plus the background and the actual detection. Including these three effects and considering the classical parametric estimation scenario we have a measurement vector $\vec{I} = (I_1, ..., I_n)$ (photoelectrons measured by the CCD) with n independent random variables driven by a Poisson distribution with expectation value given by

$$\lambda_i(F) = \mathbb{E}(I_i) = \tilde{F} \cdot g_i(x_c) + \tilde{B}_i \tag{1}$$

and

$$g_i(x_c) = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_i - \Delta x/2}^{x_i + \Delta x/2} e^{\frac{-(x-x_c)^2}{2\sigma^2}} dx \tag{2}$$

In this context, the Cramér-Rao inequality offers a lower bound for the variance of the family of unbiased estimators. More precisely, given an unbiased estimator $\widehat{\theta}_n(\cdot) : \mathbb{N}^n \to \mathbb{R}^+$ of the parameter $\theta \in \mathbb{R}^+$ to be estimated and a measurement vector $\vec{I} = (I_1, ..., I_n)$ with n independent random variables, then the Cramér-Rao bound states that:

$$\mathrm{Var}(\widehat{\theta}(I_1, ..., I_n)) \geq \frac{1}{\mathcal{I}_{\tilde{F}}(n)}, \tag{3}$$

where $\mathcal{I}_{\tilde{F}}(n)$ is the Fisher's information given by (Mendez et al. 2013):

$$\mathcal{I}_{\tilde{F}}(n) = \sum_{i=1}^{n} \left(\frac{\left(\frac{1}{\sqrt{2\pi}\sigma} \int_{x_k^-}^{x_k^+} e^{\gamma(x-x_c)} \right)^2}{\tilde{B} + \frac{\tilde{F}}{\sqrt{2\pi}\sigma} \int_{x_k^-}^{x_k^+} e^{\gamma(x-x_c)}} \right) \tag{4}$$

where $\gamma(x) \equiv \frac{1}{2}(\frac{x}{\sigma})^2$, $x_i^- = x_i - \frac{\Delta x}{2}$ and $x_i^+ = x_i + \frac{\Delta x}{2}$.

3. BAYESIAN CRAMÉR-RAO LOWER BOUND

In a Bayesian setting we assume access to a prior knowledge (Van Trees 2004; Echeverria et al. 2016) provided, for example, by stellar catalogues. It is possible to show that the mean square error (MSE) of any estimator $\hat{F}$ is bounded by:

$$\begin{aligned} \mathbb{E}[(\hat{F} - F)^2] \quad &\geq \quad \left(\mathbb{E}\left[\left(\frac{\partial \ln f_{I,F}(I, F)}{\partial F} \right)^2 \right] \right)^{-1} \\ &= \quad \frac{1}{\mathbb{E}(I_F(n)) + I(\phi)} \end{aligned} \tag{5}$$

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where $I(\phi)$ denotes the *prior information*, characterized by the probability density ϕ_F and $\mathbb{E}(I_F(n))$ is the *average* Fisher's information of the parametric setting (expectation with respect to the data). It is found that the BCR is always smaller than their parametric equivalents or Mean Cramér-Rao lower bound (MCR) (Perlman 1974; Weinstein & Weiss 1988).

$$\frac{1}{\mathbb{E}\left(I_F(n)\right)+I(\phi)} \leq \mathbb{E}\left(\frac{1}{I_F(n)}\right). \quad (6)$$

Finally, we assume for this problem of inference an unbiased Gaussian prior distribution $\phi_F = N(\mu_F, \sigma_F)$ where $\mathbb{E}\left(F\right) = \mu_F$.

4. GAIN AND ESTIMATORS

We define the gain in performance for the prior as:

$$\text{gain}(\phi) = \frac{\mathbb{E}\left(\frac{1}{I_F(n)}\right) - \frac{1}{\mathbb{E}(I_F(n))+I(\phi)}}{\mathbb{E}\left(F\right)}. \quad (7)$$

Recalling that $\sigma_m = 1.02\frac{\sigma_F}{F}$ the units for the gain (ϕ) are approximately, in magnitudes. The gain represents the improvement in photometric precision of the best estimator of the Bayes setting with respect to the best estimator of the parametric setting. In Figure 1 we evaluate the gain (ϕ) in different resolution scenarios (ultra high or survey precision) defined as $\sigma_F = \alpha \cdot \mu_F$, and $\alpha \in (0, 0.1]$ depending on the precision regime of the prior. We find that the minimum MSE is achievable by the posterior mean and it is in fact close to the BCR bound. Remarkably, when $\alpha \in (0, 0.1]$ the maximum a posteriori (MAP) decision rule is an efficient estimator that reaches the BCR (Figure 2).

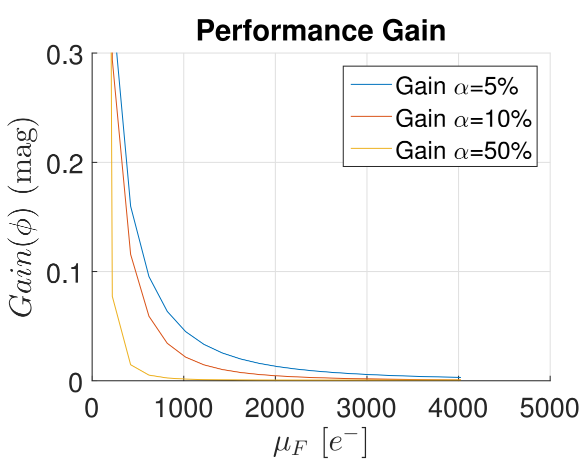

Fig. 1. Gain with respect to MCR, $B = 950$ $[e^-]$.

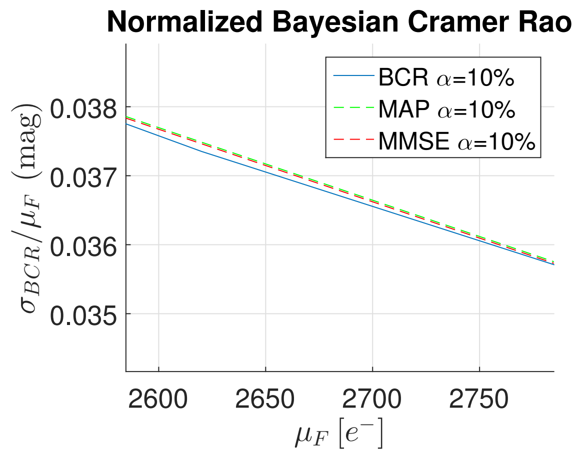

Fig. 2. Performance of Conditional Mean and MAP, $B = 950$ $[e^-]$.

5. CONCLUSIONS

We can see that the gain from the use of prior information is significant for low Signal-to-Noise regime as expected, also in the high Signal-to-Noise regime there is no appreciable gain and, hence, BCR equals the MCR.

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