Chapter 2

The Interaction of Radiation and Matter

As we saw in the previous chapter, the specific intensity is conserved along a ray unless there is the radiation interacts with matter. However, it is precisely this interaction that is the interesting part of radiation transfer and stellar atmospheres. In this chapter, we will define the quantities that describe the interaction of radiation with matter, derive the governing equations for the specific intensity and mean intensity in an atmosphere.

The Emissivity

At the macroscopic scale, emission is described by the emissivity j_ν. The energy dE added to radiation travelling through a volume dV centered on $\boldsymbol{r}$, into a solid angle $d\Omega$ centered on $\boldsymbol{n}$, in a frequency interval $(\nu, \nu + d\nu)$, and in a time interval $(t, t + dt)$ is given by

$$dE = j_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, dV\, d\Omega\, d\nu\, dt. \tag{2.1}$$

The conventional units of the j_ν are $\mathrm{erg\,s^{-1}\,Hz^{-1}\,cm^{-3}\,sr^{-1}}$. Like I_ν and I_λ, we can defined the emissivity either as j_ν per unit frequency or j_λ per unit wavelength. The two are related by $j_\nu d\nu = j_\lambda d\lambda$. We often assume that the emissivity is isotropic; the only common situations in which is it not is when there is a strong magnetic field or when we consider non-isotropic scattering. Sometimes the symbol η_ν is used for the emissivity instead of j_ν.

The Extinction Coefficient

Extincion covers the removal of energy from a ray by either absorption or scattering. At the macroscopic scale, extinction is described by the extinction coefficient χ (which is also known as the opacity or the total absorption coefficient). The energy dE removed from radiation travelling through a volume dV centered on $\boldsymbol{r}$, into a solid angle $d\Omega$ about $\boldsymbol{n}$, in a frequency interval $(\nu, \nu + d\nu)$, and in a time interval $(t, t + dt)$ is given by

$$dE = \chi(\boldsymbol{r}, \boldsymbol{n}, \nu, t) I_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, dV\, d\Omega\, d\nu\, dt. \tag{2.2}$$

The conventional units of χ are $\mathrm{cm^{-1}}$. Note that we do not use a frequency subscript in writing the extinction coefficient χ (and the related quantities α, σ, l, and τ), because χ is not defined per unit frequency, unlike I_ν, j_ν, and related quantities. In particular, the same value of the extinction coefficient is used when we are working with both I_ν and I_λ. We often assume that the extinction coefficient is isotropic; the only common situation in which is it not is when there is a strong magnetic field.

Problem 2.1 shows that the extinction coefficient χ is the inverse of the mean-free path l,

$$\chi = l^{-1}. \tag{2.3}$$

This definition of χ is for the extinction coefficient *per unit volume*. Sometimes, the extinction coefficient *per unit mass* is used to remove the direct dependence on mass density ρ. The two are related by

$$\chi^{\text{volume}} \equiv \rho\chi^{\text{mass}}. \tag{2.4}$$

We will work exclusively with the extinction coefficient per unit volume; to convert to the extinction coefficient per unit mass, one simply has to multiply all occurances of χ (and similar quantities) by the density ρ. (The emissivity j_ν is almost always considered per unit volume.)

Absorption, Emission, and Scattering

At a microscopic level, radiation interacts with matter by the processes of absorption, spontaneous emission, stimulated emission, and scattering of photons.

Absorption is perhaps the simplest process. Absorption destroys a photon and makes its energy available to the matter. This energy can change the state of the matter (provoking an excitation, ionization, or dissociation) or contribute to the kinetic energy of the matter. Examples include photoionization (bound-free transitions), line absorption (upwards bound-bound transitions), and absorption by dust grains.

Emission is the inverse process of absorption. It takes energy from the matter and creates a photon. Again, the energy can come from a change of state of the matter (a de-excitation, recombination, or association) or from the kinetic energy of the matter. Examples are recombination (free-bound transitions), line emission (downwards bound-bound transitions), and thermal emission by dust grains.

We distinguish spontaneous and stimulated emission. Spontaneous emission is straightforward, and occurs when matter in an excited state decays spontaneously, without any outside influence, and emits a photon. Stimulated emission is a curious effect that arises from photons being bosons, and thereby able to share the same quantum states. It involves the same kind of transitions as spontaneous emission, but these transition are stimulated by the presence of radiation; the probability of a spontaneous emission is independent of the radiation field, but the probability of a stimulated emission is proportional to the strength of the local radiation field.

The emissivity includes the contributions from spontaneous emission and scattering into the beam. Stimulated emission is normally considered as a negative contribution to the extinction coefficient. This may odd, but it is very convenient, as the contribution of stimulated emission to the radiation field takes the same form as absorption, being proportional to I_ν, albeit with a different sign.as the probability of stimulated emission depends on I_ν in exactly the same manner as the probability of absorption.

Scattering does not destroy or create photons, but instead changes the direction and, in some circumstances, the energy of a photon. Examples of scattering are Thomson and Compton scattering by non-relativistic and relativistic electrons, Rayleigh scattering by molecules, scattering by dust grains, and scattering by certain resonance transitions.

In the context of the description of the interaction of radiation with matter through the extinction coefficient χ and the emissivity j_ν, scattering can be regarded as the notional absorption of a photon followed instantaneously by a corresponding notional emission. We distinguish these notional absorptions and emissions from the true absorptions and emissions, arising from processes such as photoionization and recombination, which create and destroy photons. We can then write the total extinction coefficient χ as the sum of the contributions α from true absorption and σ from scattering

$$\chi = \alpha + \sigma, \tag{2.5}$$

and we can write the total emission coefficient j_ν as the sum of the contributions j_ν^{e} from true emission and j_ν^{s} from scattering,

$$j_\nu = j_\nu^{\mathrm{e}} + j_\nu^{\mathrm{s}}. \tag{2.6}$$

The quantity σ is referred to as the scattering coefficient and the quantity j_ν^{s} as the scattered emission coefficient or the scattered emissivity. The quantities α and j_ν^{e} are sometimes referred to as the true absoption coefficient and the true emissivity.

The Equation of Radiation Transfer

Derivation

Consider Figure 2.1, which shows the volume dV formed by sweeping an area dA centered on $\boldsymbol{r}$ thought a length ds parallel to $\boldsymbol{n}$ which is perpendicular to dA. Consider now the evolution

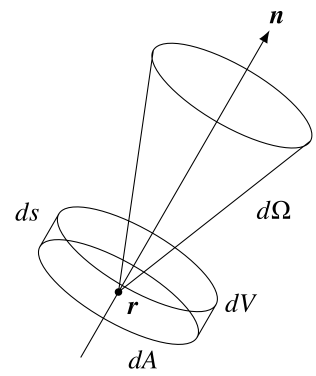

Figure 2.1: The geometry of the derivation of the transfer equation.

of the energy carried by radiation that enters the volume in solid angle $d\Omega$ centered on $\boldsymbol{n}$, in frequency interval $(\nu, \nu + d\nu)$, and in time interval $(t, d + dt)$. From the definition of specific intensity, the energy that enters is

$$I_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, dA\, d\Omega\, d\nu\, dt. \tag{2.7}$$

From the definition of the emissivity, the energy added by emission in the volume, in the same intervals of solid angle, frequency, and time, is

$$j_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, dV\, d\Omega\, d\nu\, dt. \tag{2.8}$$

From the definition of the extincion coefficient, the energy removed by extinction in volume, in the same intervals of solid angle, frequency, and time, is

$$\chi(\boldsymbol{r}, \boldsymbol{n}, \nu, t) I_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, dV\, d\Omega\, d\nu\, dt. \tag{2.9}$$

Finally, again from the definition of the specific intensity, the energy that leaves the volume, in the same intervals of solid angle and frequency but in the time interval $(t + ds/c, t + ds/c + dt)$ is

$$I_\nu(\boldsymbol{r} + \boldsymbol{n} ds, \boldsymbol{n}, \nu, t + ds/c)\, dA\, d\Omega\, d\nu\, dt. \tag{2.10}$$

The time interval is retarded by ds/c to account for the time the photons take to traverse the volume. Conservation of energy gives

$$\begin{aligned} &I_\nu(\boldsymbol{r} + \boldsymbol{n} ds, \boldsymbol{n}, \nu, t + ds/c) = \\ &I_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t) + j_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, ds - \chi(\boldsymbol{r}, \boldsymbol{n}, \nu, t) I_\nu(\boldsymbol{r}, \boldsymbol{n}, \nu, t)\, ds. \end{aligned} \tag{2.11}$$

Rearranging and letting $ds \to 0$, yields

$$\frac{\partial I_\nu}{\partial s} + \frac{1}{c}\frac{\partial I_\nu}{\partial t} = j_\nu - \chi I_\nu, \tag{2.12}$$

in which $\partial I_\nu / \partial s$ is the spatial derivative along the path of the ray. We will normally treat the case in which I_ν is independent of time, in which case we can write this equation as

$$\frac{dI_\nu}{ds} = j_\nu - \chi I_\nu. \tag{2.13}$$

Note that in the absence of interaction with matter, that is, when the emissivity and extinction coefficients are both zero, the specific intensity is conserved, as we expect.

The equation of radiation transfer becomes easier to manipulate if we change variables by introducing a quantity known as the optical depth. We can define the optical depth τ by

$$\frac{d\tau}{ds} \equiv \chi \tag{2.14}$$

along with a suitable boundary condition (e.g., $\tau = \tau_0$ at $s = 0$). With this boundary condition we can integrate this equation to give

$$\tau(s) = \tau_0 + \int_0^s ds'\ \chi(s'). \tag{2.15}$$

Since $\chi = l^{-1}$, the optical depth is also given by

$$\tau(s) = \tau_0 + \int_0^s \frac{ds'}{l(s')}, \tag{2.16}$$

and so we can see that the optical depth measures distance in units of the mean free path. The equation of radiation transfer becomes

$$\frac{dI_\nu}{d\tau} = \frac{j_\nu}{\chi} - I_\nu. \tag{2.17}$$

We can simplify this further by defining the source function S_ν as

$$S_\nu \equiv \frac{j_\nu}{\chi} = j_\nu l \tag{2.18}$$

The emissivity j_ν measures in some sense the emission per cm along the beam; since $l = \chi^{-1}$, the source function S_ν measures the emission per mean free path along the beam. However, since each mean free path is equal to a unit of optical depth, the source function S_ν also measures the emission per unit of optical depth along the beam.

In terms of the optical depth τ and the source function S_ν, equation of radiation transfer is

$$\frac{dI_\nu}{d\tau} = S_\nu - I_\nu. \tag{2.19}$$

This is the fundamental equation of radiation transfer. It is a first-order differential equation that describes how the specific intensity changes as a result of the interaction of the radiation field with matter.

It's worth considering for a moment the role of the source function in the equation of radiation transfer. If $S_\nu > I_\nu$, then $dI_\nu/d\tau$ is positive and I_ν grows along the ray; if $S_\nu < I_\nu$, then $dI_\nu/d\tau$ is negative and I_ν decreases along the ray; and if $S_\nu = I_\nu$ then $dI_\nu/d\tau$ is zero and I_ν does not change along the ray. Thus, we see that the equation of radiation transfer acts to force I_ν to approach S_ν. It does this by transferring energy either from matter to radiation or from radiation to matter, according to whether S_ν is larger or smaller than I_ν.

Formal Solution

We can obtain the formal solution to the equation of radiation transfer by moving terms in I_ν to one side and multiplying by the integrating factor e^τ, obtaining

$$e^\tau \frac{dI_\nu}{d\tau} + e^\tau I_\nu = e^\tau S_\nu. \tag{2.20}$$

This can be integrated, and the constant of integration fixed by the the boundary conditions $I_\nu(\tau_0)$, to give

$$\Big[e^t I_\nu(t)\Big]_{\tau_0}^{\tau} = \int_{\tau_0}^{\tau} dt\ e^t S_\nu(t), \tag{2.21}$$

and ultimately

$$I_\nu(\tau) = e^{(\tau_0-\tau)} I_\nu(\tau_0) + \int_{\tau_0}^{\tau} dt\ e^{(t-\tau)} S_\nu(t). \tag{2.22}$$

Thus, if we know the specific intensity of an incident ray, the value of the source function all along its path, and the value of χ all along its path (to calculate τ), we can directly calculate the specific intensity of the emergent ray.

To better understand the form of the formal result, we can make the subsitutions $\Delta\tau = \tau - \tau_0$ and $t' = \tau - t$, to give

$$I_\nu(\tau) = e^{-\Delta\tau} I_\nu(\tau_0) + \int_0^{\Delta\tau} dt'\ e^{-t'} S_\nu(\tau - t'). \tag{2.23}$$

Thus, the intensity is given by the incident intensity $I(\tau_0)$ diminished by a factor of $e^{-\Delta\tau}$, the negative exponential of the optical depth to the point of incidence, plus the integral along the path of the contributions to the intensity $S_\nu(\tau - t')$ diminished by a factor of $e^{-t'}$, again the negative exponential of the optical depth to the point making the contribution. The factor of $e^{-t'}$ is characteristic of radiation transfer problems. Mathematically, it arise because of the $-I_\nu$ term in the equation for radiation transfer. Physically, it expresses the fact that a the distance traveled by a photon has an exponential distribution, so that a fraction $1/e$ of the radiation making up the specific intensity is extinguished for each mean free path traveled. This point is explored more fully in Problem 2.1.

This expression is known as the formal solution because it is a solution for I_ν in terms of S_ν. The source function is the ratio of the emissivity j_ν and extinction coefficient χ, and so depends on the state of matter. However, as the interaction of radiation with matter determines the state of matter, the source function S_ν depends on the specific intensity I_ν. Thus, effectively, the specific intensity appears in the integral, and so the formal solution is an implicit solution.

Derivation in Plane-Parallel Symmetry

When working with plane-parallel symmetry, we often use a different definition of optical depth, which has

$$\frac{d\tau}{dz} \equiv -\chi. \tag{2.24}$$

This definition measures the optical depth parallel to the z axis, regardless of the angle of the path, and furthermore measures it into the atmosphere, so that τ increases as z decreases. Conventionally z increases outwards in the atmosphere, and so the minus sign in the definition acts so that the optical depth increases inwards. The conventional boundary condition is that $\tau = 0$ outside the atmosphere (i.e., at $z = +\infty$).

The lengths parallel to the z axis are shorter than lengths along the path by a factor of $dz/ds = \cos\theta = \mu$, where θ is the angle between the path and the z axis. Thus, we have $ds = dz/\mu = -d\tau/\mu\chi$. The equation of radiation transfer now becomes

$$\mu \frac{dI_\nu}{dz} = -\chi I_\nu + j_\nu, \tag{2.25}$$

or

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu. \tag{2.26}$$

Formal Solution in Plane-Parallel Symmetry

In plane-parallel symmetry, the formal solution becomes

$$I_\nu(\tau,\mu) = e^{(\tau-\tau_0)/\mu} I_\nu(\tau_0,\mu) + \int_\tau^{\tau_0} \frac{dt}{\mu}\, e^{(\tau-t)/\mu} S_\nu(t). \tag{2.27}$$

If we have a semi-infinite plane-parallel atmosphere (i.e., it is unbounded as $z \to -\infty$ and $\tau \to \infty$) and has no incident intensity from above (i.e., $I_\nu(0,\mu) = 0$ for $\mu \le 0$), then the intensity is given for $\mu > 0$ by

$$I_\nu(\tau,\mu) = \int_\tau^\infty \frac{dt}{\mu}\, e^{(\tau-t)/\mu} S_\nu(t), \tag{2.28}$$

and for $\mu < 0$ by

$$I_\nu(\tau,\mu) = \int_\tau^0 \frac{dt}{\mu}\, e^{(\tau-t)/\mu} S_\nu(t). \tag{2.29}$$

The first terms in the formal solution disappear for the case $\mu > 0$ as $e^{-\tau/\mu} I_\nu(\tau) \to 0$ as $\tau \to \infty$ for any finite $I_\nu(\tau)$. It disappears for the case $\mu < 0$ by the boundary condition $I_\nu(0,\mu) = 0$ for $\mu < 0$.

These results again become clearer and more symmetrical if we make the substitutions $t' = t - \tau$ for $\mu > 0$ and $t' = \tau - t$ for $\mu < 0$, and also use $|\mu|$ to avoid confusion over the sign of μ. The formal solutions is then given by

$$I_\nu(\tau,\mu) = \begin{cases} \int_0^\infty \frac{dt'}{|\mu|}\, e^{-t'/|\mu|} S_\nu(\tau + t') & \text{for } \mu > 0, \\ \int_0^\tau \frac{dt'}{|\mu|}\, e^{-t'/|\mu|} S_\nu(\tau - t') & \text{for } \mu < 0. \end{cases} \tag{2.30}$$

Thus, the intensity is given by the integral along the path to the appropriate boundary of the contribution to the intensity S_ν diminished by a factor $e^{-t'/|\mu|}$.

An important special case is the emergent intensity $I_\nu(0,\mu)$ for $\mu > 0$ from a semi-infinite atmosphere, which is given by

$$I_\nu(0,\mu) = \int_0^\infty \frac{dt}{\mu}\, e^{-t/\mu} S_\nu(t). \tag{2.31}$$

Note that $I_\nu(0,\mu)$ is the Laplace transform of $S_\nu(\mu t)$; this can be useful sometimes in understanding the dependence of the emergent flux on the source function and, occasionally, in seeking solutions. The emergent flux $F_\nu(0)$ is given by

$$F_\nu(0) = 2\pi \int_0^1 d\mu \int_0^\infty dt\, e^{-t/\mu} S_\nu(t). \tag{2.32}$$

The Schwarzschild Equation

We can write the mean intensity $J_\nu(\tau)$ in terms of the upwards and downwards mean intensities $J_\nu^+(\tau)$ and $J_\nu^-(\tau)$ as

$$J_\nu(\tau) = \frac{1}{2}\left[J_\nu^+(\tau) + J_\nu^-(\tau)\right], \tag{2.33}$$

where

$$J_\nu^+(\tau) \equiv \int_0^{+1} d\mu\, I_\nu(\tau,\mu), \tag{2.34}$$

and

$$J_\nu^-(\tau) \equiv \int_{-1}^0 d\mu\, I_\nu(\tau,\mu). \tag{2.35}$$

We can use the formal solution for $I_\nu(\tau,\mu)$ in these definitions to obtain

$$J_\nu^+(\tau) = \int_0^{+1} d\mu \int_\tau^\infty \frac{dt}{\mu}\, e^{(\tau-t)/\mu} S_\nu(t), \tag{2.36}$$

and

$$J_\nu^-(\tau) = \int_{-1}^0 d\mu \int_\tau^0 \frac{dt}{\mu}\, e^{(\tau-t)/\mu} S_\nu(t). \tag{2.37}$$

If we make the substitutions $w = +\mu^{-1}$ in the expression for J_ν^+ and $w = -\mu^{-1}$ in the expression for J_ν^-, we obtain

$$J_\nu^+(\tau) = \int_\tau^\infty dt\, S_\nu(t) \int_1^\infty \frac{dw}{w} e^{w(\tau-t)}, \tag{2.38}$$

and

$$J_\nu^-(\tau) = \int_0^\tau dt\, S_\nu(t) \int_1^\infty \frac{dw}{w} e^{w(t-\tau)}. \tag{2.39}$$

The integrals over w have the form of the first exponential integral E_1 (Abramowitz & Stegun 1972, pp. 228-231), where E_n is defined by

$$E_n(x) \equiv \int_1^\infty dw\, w^{-n} e^{-xw}. \tag{2.40}$$

Thus

$$J_\nu^+(\tau) = \int_\tau^\infty dt\ S_\nu(t)E_1(t-\tau), \tag{2.41}$$

and

$$J_\nu^-(\tau) = \int_0^\tau dt\ S_\nu(t)E_1(\tau-t). \tag{2.42}$$

However, given the ranges of the two integrals over t, in both expressions the argument to the exponential integral is non-negative, so we can write it as $|t-\tau|$. After this, the two integrands are identical, and we can combine the integrals to give

$$J_\nu(\tau) = \frac{1}{2}\int_0^\infty dt\ S_\nu(t)E_1(|t-\tau|). \tag{2.43}$$

This is known as the Schwartzschild equation Schwarzschild (1914).

For conciseness, we can define an operator Λ_1, such that

$$\Lambda_1(f) \equiv \frac{1}{2}\int_0^\infty dt\ f(t)E_1(|t-\tau|). \tag{2.44}$$

With this notation, the Schwartzschild equation becomes

$$J_\nu(\tau) = \Lambda_1(S_\nu). \tag{2.45}$$

The Λ operator is almost ubiquitous in the development of numerical solutions to the equation of radiative transfer.

The Milne Equation

In a similar manner to the Schwarzschild equation, we can obtain the Milne equation (Milne 1930, p. 130),

$$H_\nu(\tau) = \frac{1}{2}\int_\tau^\infty dt\ S_\nu(t)E_2(t-\tau) - \frac{1}{2}\int_0^\tau dt\ S_\nu(t)E_2(\tau-t) \tag{2.46}$$

$$\equiv \Lambda_2(S_\nu). \tag{2.47}$$