

Chapter 3

Simplifying Approximations

In this chapter we will introduce a number of widely-used approximations that can simplify considerably the solution of the equation of radiation transfer in atmospheres.

Coherent and Isotropic Scattering

Scattering is easy to incorporate into extinction – we simply include its contribution to the extinction coefficient – but a general treatment of the scattered emissivity is somewhat involved; Schuster (1905, p. 5), in his pioneering investigation of scattering in stellar atmospheres, notes that “the complete investigation leads to equations of such complexity that a discussion becomes impossible”.

Given this, we will make two simplifying assumptions. First, we will assume that the scattering is coherent, so that the scattering does not change the frequency of the photon. Second, we will assume that the scattering is isotropic, so that photons are emitted with equal probability into all directions. Under these assumptions, the energy removed from the radiation field by scattering in a volume dV , a frequency interval $(\nu, \nu + d\nu)$, and a time interval $(t, t + dt)$ must be equal to the energy added to the radiation field by scattering in the same volume dV , the same frequency interval $(\nu, \nu + d\nu)$, and the same time interval $(t, t + dt)$. From the definition of the extinction coefficient and the emissivity, we then have

$$\int_{4\pi} d\Omega \sigma I_\nu = \int_{4\pi} d\Omega j_\nu^s \quad (3.1)$$

$$4\pi\sigma J_\nu = 4\pi j_\nu^s. \quad (3.2)$$

Thus, the scattered emissivity is given by

$$j_\nu^s = \sigma J_\nu. \quad (3.3)$$

Local Thermodynamic Equilibrium

In general, to determine the true emissivity j_ν^e and the absorption coefficient α , we need to determine the state of matter in the atmosphere. As we will see, this is often quite

difficult because of the coupling between radiation and matter. For this reason, we often make adopt the simplifying assumption of local thermodynamic equilibrium (LTE).

In perfect thermodynamic equilibrium, we have detailed balance in all processes. Detailed balance means that the rate of a process and its inverse are equal, and follows from the symmetry of the laws of microscopic physics under time reversals. If we consider absorption and emission, in perfect thermodynamic equilibrium we have

$$j_\nu^e = \alpha I_\nu. \quad (3.4)$$

However, we also have that $I_\nu = B_\nu$, and so we derive Kirchoff’s law, that in perfect thermodynamic equilibrium

$$j_\nu^e = \alpha B_\nu(T). \quad (3.5)$$

It’s important to remember that is true only in perfect thermodynamic equilibrium, and that stellar atmospheres are *not* in perfect thermodynamic equilibrium (as witnessed by, for example, the flow of energy with them).

In the approximation of local thermodynamic equilibrium (LTE), we assume that the matter has thermodynamic equilibrium properties at the local temperature and density but the radiation does not. In particular, the velocities of particles is assumed to be given by the Maxwell distribution, the abundances of chemical species is assumed to be given by the Saha distribution, and the populations of energy levels is assumed to be given by the Boltzmann distribution, but we do *not* assume that the specific intensity is given by the Planck function. The emissivity and extinction coefficient are determined by the state of matter, so Kirchoff’s law will still apply in LTE, and we still have

$$j_\nu^e = \alpha B_\nu(T). \quad (3.6)$$

It is vital to remember that these results are not true in general; in order to obtain them, we had to assume LTE. Radiation emitted by matter in LTE (or, more accurately, matter that is well-approximated by the LTE approximation) is known as thermal radiation.

LTE significantly simplifies the solution of the equation transfer, in that it reduces the state of matter at each point

to two variables – the temperature and the density. Nevertheless, the problem is still complicated, as the temperature is determined by the interaction of radiation with matter.

Note that LTE is an internally inconsistent approximation; the occupation numbers will only have their true thermal equilibrium values if the temperature is uniform and the radiation field isotropic and Planckian. Both of these requirements are violated in stellar atmospheres, in which we often have large temperature gradients and in which the radiation field is sharply peaked in the outward direction and is non-Planckian. The LTE approximation is not too bad for the velocity distribution of particles; collisions are sufficiently frequent to maintain a single-temperature Maxwellian distribution. However, it is much worse for the ionization and excitation distributions. To be fully consistent, we would have to adopt a completely non-LTE approach and directly model the processes that create and destroy chemical species and populate and depopulate energy levels. We will use this approach in the later parts of this book, but for the meantime we will adopt the LTE approximation.

In LTE we have $j_\nu^e = \alpha B_\nu$, and so in general the source function is

$$S_\nu \equiv \frac{j_\nu}{\chi} = \frac{\alpha B_\nu + j_\nu^s}{\alpha + \sigma}. \quad (3.7)$$

Two important special cases of this are the absence of scattering, which has $j_\nu^s = 0$ and

$$S_\nu = B_\nu, \quad (3.8)$$

and coherent, isotropic scattering, which has $j_\nu^s = \sigma J_\nu$ and

$$S_\nu = \frac{\alpha B_\nu + \sigma J_\nu}{\alpha + \sigma}. \quad (3.9)$$

Radiative Equilibrium

Stellar atmospheres are often assumed to be in thermal equilibrium. By this we mean that the temperature at a given point is constant in time,

$$\frac{\partial T(\mathbf{r})}{\partial t} = 0. \quad (3.10)$$

In the plane-parallel approximation, this requires that the total flux of energy be conserved throughout the atmosphere. A special case of thermal equilibrium is *radiative equilibrium*, in which the flux of energy is carried entirely by radiation. In this case, in the plane-parallel approximation, F is constant throughout the atmosphere, or in other words $dF/dz = 0$. The same radiative flux is found entering the atmosphere from below, leaving it to above, and in all intermediate layers.

The constancy of F is a global statement of the condition of radiative equilibrium in a plane-parallel atmosphere. We

can also devise a local statement: that the radiative energy absorbed in a small volume must be equal to the radiative energy emitted by the same volume. If this were not the case, the volume would suffer a net gain or loss of energy and would rise or fall in temperature. From the definition of the extinction coefficient, we can see that the total radiative heating rate per unit volume is

$$\int_0^\infty dv \int_{4\pi} d\Omega \chi I_\nu = 4\pi \int_0^\infty dv \chi J_\nu, \quad (3.11)$$

where in substituting J_ν we have assumed that χ is isotropic. Similarly, from the definition of the emissivity, the total radiative cooling rate per unit volume is

$$\int_0^\infty dv \int_{4\pi} d\Omega j_\nu = 4\pi \int_0^\infty dv j_\nu, \quad (3.12)$$

where again we have assumed that j_ν is isotropic. In radiative equilibrium, we have the local condition

$$\int_0^\infty dv \chi J_\nu = \int_0^\infty dv j_\nu. \quad (3.13)$$

Note that in deriving the local condition we have not made any assumption about the geometry of the atmosphere. Thus, the local condition holds in any atmosphere in radiative equilibrium, whereas the global condition that F is constant is true only for a plane-parallel atmosphere in radiative equilibrium. For example, in a spherically symmetric atmosphere in radiative equilibrium, F drops as $1/r^2$.

At first glance, it seems odd that in a plane-parallel atmosphere we have two statements of the condition for radiative equilibrium: $dF/d\tau = 0$ and $\int_0^\infty dv \chi J_\nu = \int_0^\infty dv j_\nu$. However, as we might expect, the two are equivalent, and each implies the other. This is explored in Problem 3.1.

The Eddington-Barbier Approximations

The Eddington-Barbier relations are the basis of a very useful approximation. If we assume that we can approximate the source function by a linear function in τ , that is

$$S_\nu(\tau) \approx a_\nu + b_\nu \tau, \quad (3.14)$$

then by the Eddington-Barbier relations we have

$$I_\nu(0, \mu) \approx S_\nu(\tau = \mu), \quad (3.15)$$

and

$$F_\nu(0) \approx \pi S_\nu(\tau = 2/3). \quad (3.16)$$

These approximate results are known as the Eddington-Barbier approximations.

It may seem pedantic to distinguish between the Eddington-Barbier relations, which are exact results for exactly linear source functions, and the Eddington-Barbier approximations, which are approximate results for approximately linear source functions. However, it is important to remember that the degree to which the approximate results are good depends on the degree to which the underlying approximation about the source function is a good approximation.

The Eddington-Barbier approximations are extremely useful in considering the qualitative behaviour of stellar atmospheres. For example, we can determine $I(0, \mu)$ by measuring the emergent intensity of the Sun at different angles, i.e., by measuring the emergent intensity from the center of its disk ($\mu = 1$) to its limb ($\mu \approx 0$). The Sun is the only star which is sufficiently resolved to allow us to measure $I(0, \mu)$ directly in this manner. If we then adopt the Eddington-Barbier approximation, we can obtain the approximate values of S_ν in the range $0 \leq \tau \leq 1$. If we are prepared to assume LTE and ignore scattering, we can relate this directly to the local temperature, because in LTE $S_\nu = B_\nu(T)$.

If we combine the LTE approximation with the Eddington-Barbier approximation, we can determine the temperature in the atmosphere of the Sun from observations of the limb darkening, as we have

$$I_\nu(0, \mu) = S_\nu(\tau = \mu) = B_\nu(T(\tau = \mu)). \quad (3.17)$$

This can currently only be applied to the Sun, as this is the only star for which we have accurate measurements of $I(0, \mu)$ (i.e., it is the only star that is well resolved). Problem 3.2 investigates this in more detail.

Notes and Further Reading

The Eddington-Barbier Approximation

The origins of the use of the Eddington-Barbier approximation for determining χ from the solar limb darkening function (see problem 3.2) lie in investigations by Milne (1922) and Lundblad (1923). A determination of the empirical extinction coefficient by Münch (1945) and Chandrasekhar & Münch (1946), using a more sophisticated method and better observations, showed good agreement with calculations by Chandrasekhar & Breen (1946) of opacity of the H^- ion, and confirmed the suggestion made by Wildt (1939) that H^- was the dominant source of opacity in the Sun at optical and near-infrared wavelengths. Mihalas (1978, pp. 260–263) discusses the limitations imposed by observational uncertainties.