Chapter 4

Simple Applications of Radiation Transfer

The Diffusion Approximation

Statement of Problem

The diffusion approximation is an approximate solution to the radiation transfer equation under that assumptions of plane-parallel symmetry, that matter is in LTE, scattering is coherent and isotropic, that the temperature gradient $dT/d\tau$ is small, and the optical depth to the surface is large.

These last two assumptions are never satisfied in the atmosphere of a star; indeed, the atmosphere is characterised by large temperature gradients and by proximity to the surface. However, they are very good approximations deep in the star at large optical depth. The diffusion approximation is useful for modelling the radiative flux or energy in the stellar interior and for providing a lower boundary condition for stellar atmosphere models.

Solution

If we have LTE and coherent and isotropic scattering, we have

$$S_\nu = \frac{\alpha B_\nu + \sigma J_\nu}{\alpha + \sigma}. \tag{4.1}$$

Therefore, S_ν is isotropic. We can then expand S_ν as a McLaurin series in τ to obtain

$$S_\nu(\tau + t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{d^n S_\nu}{d\tau^n} \tag{4.2}$$

If we substitute this into the formal solution for I_ν in $0 \leq \mu \leq 1$, we obtain

$$\begin{aligned} I_\nu(\mu) &= \int_0^\infty \frac{dt}{\mu}\, e^{-t/\mu} S(\tau + t) && (4.3) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n S_\nu}{d\tau^n} \int_0^\infty \frac{dt}{\mu}\, e^{-t/\mu} t^n. && (4.4) \end{aligned}$$

If we make the substitution $x = t/\mu$, we obtain

$$I_\nu(\mu) = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \frac{d^n S_\nu}{d\tau^n} \int_0^\infty dx\, e^{-x} x^n. \tag{4.5}$$

The integrals $\int_0^\infty dx\, e^{-x} x^n \equiv \Gamma(n+1)$, are pure numbers, and we can obtain their values by induction. For $n > 0$ we can integrate $\Gamma(n+1)$ by parts to give

$$\Gamma(n+1) = \left[-e^{-x} x^n\right]_0^\infty + n \int_0^\infty dx\, e^{-x} x^{n-1}. \tag{4.6}$$

The first term is zero and the second term is just $n\Gamma(n)$, so we have

$$\Gamma(n+1) = n\Gamma(n). \tag{4.7}$$

Furthermore, we have

$$\Gamma(1) = \int_0^\infty dx\, e^{-x} = \left[-e^{-x}\right]_0^\infty = 1. \tag{4.8}$$

Thus, by induction,

$$\Gamma(n+1) = n!. \tag{4.9}$$

The Γ function is one of the common mathematical functions of physics; is simply a generalization of the familiar factorial function for non-integer arguments. With these values of $\Gamma(n+1)$, we obtain for $0 \leq \mu \leq 1$,

$$I_\nu(\mu) = \sum_{n=0}^{\infty} \mu^n \frac{d^n S_\nu}{d\tau^n}. \tag{4.10}$$

The expression for $I_\nu(\mu)$ for $-1 \leq \mu \leq 0$ differs only by terms of order $e^{-\tau/\mu}$, which tend to zero as τ becomes larger. Thus, we can use this expression as an approximation for I_ν for all μ, obtaining

$$I_\nu(\mu) \approx \sum_{n=0}^{\infty} \mu^n \frac{d^n S_\nu}{d\tau^n}. \tag{4.11}$$

Putting this approximation for I_μ into the definitions of

J_ν, we obtain

$$J_\nu \approx \frac{1}{2}\int_{-1}^{+1} d\mu \sum_{n=0}^{\infty} \frac{d^n S_\nu}{d\tau^n}\mu^n \tag{4.12}$$
$$\approx \frac{1}{2}\sum_{n=0}^{\infty} \frac{d^n S_\nu}{d\tau^n}\int_{-1}^{+1} d\mu\, \mu^n \tag{4.13}$$
$$\approx \frac{1}{2}\sum_{n=0}^{\infty} \frac{d^n S_\nu}{d\tau^n}\left[\frac{\mu^{n+1}}{n+1}\right]_{-1}^{+1} \tag{4.14}$$
$$\approx \sum_{n=0}^{\infty} \frac{1}{2n+1}\frac{d^{2n} S_\nu}{d\tau^{2n}} \tag{4.15}$$
$$\approx S_\nu + \frac{1}{3}\frac{d^2 S_\nu}{d\tau^2} + \frac{1}{5}\frac{d^4 S_\nu}{d\tau^4} + \cdots. \tag{4.16}$$

Notice that the sum only contains terms in even derivatives of S_ν. We now use our assumption that the temperature gradients are small. In this case, to a good approximation we can to truncate the series after the first term, to give

$$J_\nu \approx S_\nu. \tag{4.17}$$

We still do not know S_ν. However, if we substitute $J_\nu \approx S_\nu$ into equation 4.1, we obtain

$$S_\nu \approx \frac{\alpha B_\nu + \sigma S_\nu}{\alpha + \sigma}, \tag{4.18}$$

which has the solution

$$S_\nu \approx B_\nu. \tag{4.19}$$

We now know the source function; it is simply the Plank function at the local temperature.

We can now calculate other useful quantities. For example, the energy flux F_ν is given by

$$F_\nu(\tau) = 2\pi\int_{-1}^{+1} d\mu\, \mu I_\nu \tag{4.20}$$
$$\approx 2\pi\sum_{n=0}^{\infty}\int_{-1}^{+1} d\mu\, \mu^{n+1}\frac{d^n S_\nu}{d\tau^n} \tag{4.21}$$
$$\approx 2\pi\sum_{n=0}^{\infty}\frac{d^n B_\nu}{d\tau^n}\left[\frac{\mu^{n+2}}{n+2}\right]_{-1}^{+1} \tag{4.22}$$
$$\approx 4\pi\sum_{n=0}^{\infty}\frac{1}{2n+3}\frac{d^{2n+1} B_\nu}{d\tau^{2n+1}} \tag{4.23}$$
$$\approx \frac{4\pi}{3}\frac{dB_\nu}{d\tau} + \frac{4\pi}{5}\frac{dB_\nu^3}{d\tau^3} + \ldots. \tag{4.24}$$

Again, since $dB_\nu/d\tau$ is small, we can ignore all but the leading term and obtain

$$F_\nu(\tau) \approx \frac{4\pi}{3}\frac{dB_\nu}{d\tau} \tag{4.25}$$

If we use the chain rule, we can write

$$\frac{dB_\nu}{d\tau} = \frac{dB\nu}{dT}\frac{dT}{dz}\frac{dz}{d\tau} \tag{4.26}$$
$$= -\frac{1}{\chi}\frac{dB\nu}{dT}\frac{dT}{dz}, \tag{4.27}$$

and so obtain

$$F_\nu(\tau) = -\left[\frac{4\pi}{3}\frac{1}{\chi}\frac{dB_\nu}{dT}\right]\frac{dT}{dz}. \tag{4.28}$$

If we integrate over all frequencies, we obtain

$$F = -\left[\frac{4\pi}{3}\int_0^\infty d\nu \frac{1}{\chi}\frac{dB_\nu}{dT}\right]\frac{dT}{dz} \tag{4.29}$$
$$= -\left[\frac{4\pi}{3}\frac{1}{\chi_\mathrm{R}}\frac{dB}{dT}\right]\frac{dT}{dz}, \tag{4.30}$$

where the Rosseland mean opacity χ_R is defined by

$$\frac{1}{\chi_\mathrm{R}}\frac{dB}{dT} \equiv \int_0^\infty d\nu\frac{1}{\chi}\frac{dB_\nu}{dT}. \tag{4.31}$$

This equation has the form of a diffusion equation,i.e.,

$$\text{flux} = -\text{diffusion coefficient} \times \text{gradient}. \tag{4.32}$$

From this the term “diffusion approximation” is derived.

Quick Solution

We can quickly obtain the approximate result for the flux in the diffusion approximation by using the Eddington-Barbier approximation. First of all, we assume that the gradients are small, so we can approximate S_ν by the first two terms of the McLaurin expansion,

$$S_\nu(\tau + t) \approx S_\nu(\tau) + t\frac{dS_\nu(\tau)}{d\tau}. \tag{4.33}$$

If we consider a notional surface at τ bounding the matter from above and use the Eddington-Barbier approximation, we have for $\mu > 0$,

$$I_\nu(\tau,\mu) \approx S_\nu(\tau+\mu) \tag{4.34}$$
$$\approx S_\nu(\tau) + \mu\frac{dS_\nu(\tau)}{d\tau}. \tag{4.35}$$

For $\mu < 0$, we can again consider a notional surface at τ bounding the matter from below and assume that τ is large. Doing so, we obtain the same relation for I_ν. Thus, for all μ,

$$I_\nu(\tau,\mu) \approx S_\nu(\tau) + \mu\frac{dS_\nu(\tau)}{d\tau}. \tag{4.36}$$

We now integrate over angle to obtain J_ν. The term in μ is odd and cancels, leaving us with

$$J_\nu = S_\nu \tag{4.37}$$

Under LTE and with coherent and isotropic scattering, we have

$$S_\nu \approx \frac{\alpha B_\nu + \sigma J_\nu}{\alpha + \sigma}, \tag{4.38}$$

and substituting for J_ν we obtain

$$S_\nu \approx B_\nu. \tag{4.39}$$

We divide the flux into two parts, $+F_\nu^+$ being contributed by rays with $\mu > 0$ and $-F_\nu^-$ being contributed by rays with $\mu < 0$. Thus,

$$F_\nu = F_\nu^+ - F_\nu^- \tag{4.40}$$

Again, using the Eddington-Barbier approximation, we have

$$F_\nu^+ \approx \pi S_\nu(\tau + 2/3) \tag{4.41}$$

$$\approx \pi \left[B_\nu(\tau) + \frac{2}{3}\frac{dB_\nu(\tau)}{d\tau} \right] \tag{4.42}$$

and

$$F_\nu^- \approx \pi S_\nu(\tau - 2/3) \tag{4.43}$$

$$\approx \pi \left[B_\nu(\tau) - \frac{2}{3}\frac{dB_\nu(\tau)}{d\tau} \right]. \tag{4.44}$$

We then have

$$F_\nu \approx \frac{4\pi}{3}\frac{dB_\nu(\tau)}{d\tau} \tag{4.45}$$

Application

The diffusion approximation is used deep in a star to describe the flow of energy carried by radiation. However, in an atmosphere the optical depth to the surface is not small, and so the diffusion approximation is not valid. If we use the diffusion approximation as a lower boundary condition for an atmosphere, we must place the lower boundary at a depth that is sufficiently large, typically $\tau_R = 10$ to $\tau_R = 20$.

The Grey Atmosphere

Statement of Problem

The problem of the the grey atmosphere consists of solving the equations of radiative transfer under the assumptions that the opacity is independent of wavelength, the source function is isotropic, and the atmosphere is in radiative equilibrium. In order to derive the frequency dependence of quanties, we shall also later assume LTE.

When a quantity does not depend on wavelength, we say it is "grey". The assumption that the opacity is grey is a good approximation for completely ionized gases in which the only source of opacity is electron scattering, but is obviously less good in partially ionized gases, in which bound-bound and bound-free transitions contribute to make the opacity non-grey. Nevertheless, the grey atmosphere is a useful starting place for interative solutions and a useful didactic tool for understanding the qualitative behaviour of stellar atmospheres.

Development of Solution

The equation of transfer is

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu. \tag{4.46}$$

Integrating over wavelength gives us,

$$\mu \int_0^\infty d\nu \, \frac{dI_\nu}{d\tau} = I - S. \tag{4.47}$$

Since the opacity is grey, the optical depth is grey too. Thus, we can take the derivative with respect to τ outside of the integral and obtain

$$\mu \frac{dI}{d\tau} = I - S. \tag{4.48}$$

Several useful results can be obtained by considering the moments of the integrated radiation transfer equation. Taking the first moment gives

$$M_0(\mu \frac{dI}{d\tau}) = M_0(I) - M_0(S), \tag{4.49}$$

$$\frac{dM_0(\mu I)}{d\tau} = J - S, \tag{4.50}$$

$$\frac{dH}{d\tau} = J - S, \tag{4.51}$$

in which we have assumed that S is isotropic. However, as we have assumed radiative equilibrium, the total Eddington flux H must be a constant, so $dH/d\tau = 0$, and we have

$$J = S. \tag{4.52}$$

We could also have obtained this by considering the local condition for radiative equilibrium,

$$\int_0^\infty d\nu \chi J_\nu = \int_0^\infty d\nu j_\nu, \tag{4.53}$$

where we can move χ from one integral to the other, giving $J = S$.

This simple relation for the integrated source function is a vital simplification, as it allows us to eliminate the source function. The Schwarzschild equation, for example, becomes

$$J = \Lambda_\tau(J), \tag{4.54}$$

which is a linear integral equation in J only. Despite this simplification, direct solution of the Schwarzschild equation is still challenging. However, we can obtain the form of the solution by considering the first moment of the integrated transfer equation,

$$M_1(\frac{\mu dI}{d\tau}) = M_1(I) - M_1(S), \tag{4.55}$$

which, as S is again assumed to be isotropic, simplifies to

$$\frac{dK}{d\tau} = H. \tag{4.56}$$

However, since H is constant, we can integrate this directly to give

$$K = H(\tau + c), \tag{4.57}$$

where c is a constant of integration. Unfortunately, we desire S or J rather than K. We can relate K and J through the Eddington factor $f \equiv K/J$ as

$$J = K/f = H(\tau + c)/f, \tag{4.58}$$

which appears to be nothing more than a recasting of our ignorance of J as an ignorance of f. However, deep in the atmosphere the diffusion approximation will be valid, the radiation field will tend to isotropy, and the Eddington factor will tend to 1/3 (see Problem 4.1). Thus, as $\tau \to \infty$, $J \to 3K = 3H\tau$. This suggests that the J will have the form

$$J = 3H(\tau + q(\tau)), \tag{4.59}$$

where $q(\tau)$ remains bounded as $\tau \to \infty$. The function $q(\tau)$ is known as the Hopf function.

Eddington Approximate Solution

One approximate solution for $q(\tau)$ comes from making what is known generally as the Eddington approximation: that the Eddington factor $f \equiv K/J$ is 1/3 everywhere. (The Eddington factor is only exactly 1/3 for an isotropic radiation field.) From our expressions for J and K, we see that in general

$$f \equiv \frac{K}{J} = \frac{1}{3}\frac{\tau + c}{\tau + q(\tau)}. \tag{4.60}$$

The only way that we can have $f = 1/3$ everywhere is if the Hopf function $q(\tau)$ is given by

$$q(\tau) = c. \tag{4.61}$$

To obtain the value of c, we consider the flux at the surface. As the Hopf function is constant, J is a linear function in τ, and since $J = S$, we can use the Eddington-Barbier relation to obtain

$$F(0) = \pi S(\tau = 2/3) = 3\pi H(2/3 + c). \tag{4.62}$$

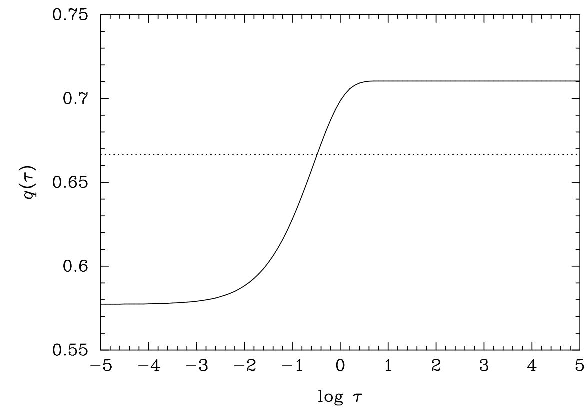

Figure 4.1: The value of $q(\tau)$ for the Eddington approximate solution (dotted line) and the exact solution (dashed line).

However, because of radiative equilibrium, the flux is constant and has the value $F = 4\pi H$. Thus,

$$4\pi H = 3\pi H(2/3 + c), \tag{4.63}$$

and so $c = 2/3$. With this, we see that the Eddington approximate solution is

$$J = 3H(\tau + 2/3). \tag{4.64}$$

The Eddington approximate solution is remarkably good, considering the rather crude assumptions from which it stems. As might be expected, it is worst close to the surface, where the radiation field becomes forward peaked and the Eddington factor f rises above 1/3. It is easy to show that close to the surface the Eddington approximate solution is not self-consistent. We can use the source function to calculate $I(\tau, \mu)$ and thus $J(\tau)$ and $K(\tau)$. These reveal inconsistencies; for that the calculated $J(\tau)$ is not equal to $S(\tau)$ and that the calculated Eddington factor is not equal to 1/3. This is demonstrated at the surface in Problem 4.3. Despite these problems, the Eddington approximation is very useful as a tool to understand grey atmospheres and as a starting point for more accurate solutions.

Exact Solution

The exact solution for $q(\tau)$ was first obtained by Mark (1947) in the context of neutron scattering. There are a variety of approaches to and forms of the solution (Chandrasekhar 1960, ch. 4 and 5; Kourganoff 1952, §27, §28, and §29; Woolley & Stibbs 1953, ch. 3), but a form that is convenient for numerical computation is

$$q(\tau) = q(\infty) - \frac{1}{2\sqrt{3}}\int_0^1 du \frac{e^{-\tau/u}}{H(u)Z(u)} \tag{4.65}$$

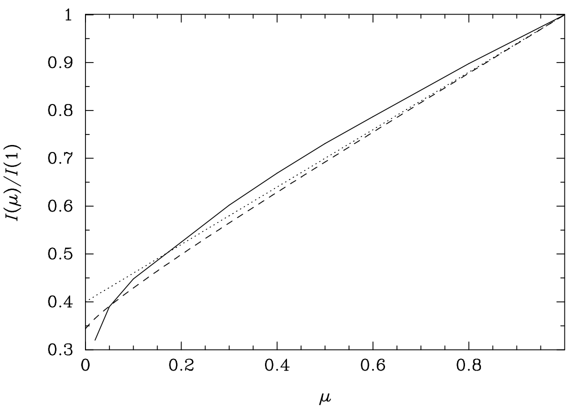

Figure 4.2: The observed solar limb darkening (solid line) $I(\mu)/I(1)$ compared to the predictions of the Eddington approximate solution to the grey atmosphere (dotted line) and the exact solution to the grey atmosphere (dashed line).

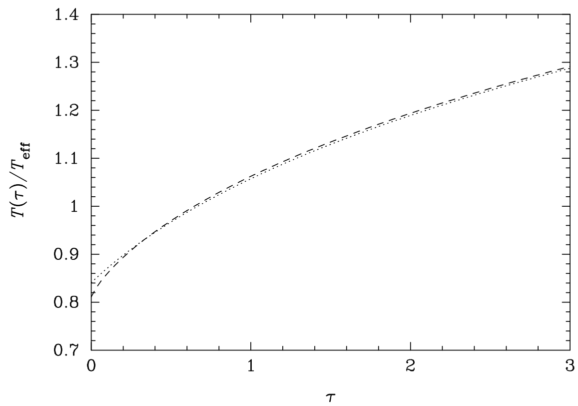

Figure 4.3: The value of $T(\tau)/T_{\rm eff}$ for the Eddington approximate solution to the grey atmosphere (dotted line) and the exact solution to the grey atmosphere (dashed line).

where

$$q(\infty) = \frac{\int_0^1 du\, H(u)u^2}{\int_0^1 du\, H(u)u}, \tag{4.66}$$

$$H(u) \equiv \frac{\exp\left[\frac{1}{\pi}\int_0^{\pi/2} d\theta\, \frac{\theta \arctan(u \tan\theta)}{1-\theta\cot\theta}\right]}{\sqrt{1+u}}, \tag{4.67}$$

$$Z(u) \equiv \left[1 - \frac{u}{2}\ln\left(\frac{1+u}{1-u}\right)\right]^2 + \frac{1}{4}\pi^2 u^2. \tag{4.68}$$

The form of this expression for $q(\tau)$ is a strong indication that the derivations are anything but trivial. (The function $H(\mu)$ should not be confused with the Eddington flux H.) Figure 4.1 shows the value of $q(\tau)$ for the Eddington approximate solution and for the exact solution. Note that most of the change in the exact $q(\tau)$ occurs at small optical depths, and that the Eddington approximate value for $q(\tau)$ lies between the two exact limiting cases of $q(0) = 1/\sqrt{3} = 0.577$ and $q(\infty) = 0.710$.

Limb Darkening

The source function for the Eddington approximate solution is linear in τ, so we can also use the Eddington-Barbier relation to give the emergent integrated intensity,

$$I(0,\mu) = 3H(\mu + 2/3). \tag{4.69}$$

Which predicts that the integrated limb darkening is given by

$$\frac{I(0,\mu)}{I(0,1)} = \frac{2/3+\mu}{2/3+1} = \frac{2+3\mu}{5}. \tag{4.70}$$

We can't use the Eddington-Barbier relation for the exact solution, as the source function is no longer linear in τ, but the development of the exact solution gives

$$\frac{I(0,\mu)}{I(0,1)} = \frac{H(\mu)}{H(1)}. \tag{4.71}$$

Figure 4.2 compares the limb darkening for the Sun, for the Eddington approximate solution, and for the exact solution. The agreement is perhaps surprisingly good, given that the solar atmosphere is certainly not grey. Schwarzschild (1906) obtained a similarly good agreement between a slightly different approximate solution for the grey atmosphere and observations, and it was precisely this that lead him to suggest that the photosphere of the Sun was in radiative equilibrium rather than convective equilibrium.

Temperature

Until now, we have been working with frequency-integrated quantities, such as F. To make progress with the frequency-dependent quantities, such as F_ν, we need the frequency dependence of the source function. This can be easily obtained if we are prepared to make the assumption of LTE and coherent and isotropic scattering. We then have

$$S_\nu = \frac{\alpha B_\nu + \sigma J_\nu}{\alpha + \sigma}, \tag{4.72}$$

but since α and σ are grey, we can integrate this over frequency to give

$$S = \frac{\alpha B + \sigma J}{\alpha + \sigma}. \tag{4.73}$$

We also know that in the grey atmosphere

$$S = J, \tag{4.74}$$

and so we have

$$J = \frac{\alpha B + \sigma J}{\alpha + \sigma}. \tag{4.75}$$

The solution to this is

$$S = J = B. \tag{4.76}$$

Substituting for $S(\tau)$, we find

$$S(\tau) = 3H(\tau + q(\tau)) = B(\tau) = \frac{\sigma}{\pi}T^4(\tau), \tag{4.77}$$

and, by the definition of $T_{\rm eff}$,

$$H = \frac{F}{4\pi} = \frac{\sigma T_{\rm eff}^4}{4\pi}. \tag{4.78}$$

Thus,

$$T^4(\tau) = \frac{3}{4}T_{\rm eff}^4(\tau + q(\tau)). \tag{4.79}$$

Figure 4.3 shows the temperature $T(\tau)$ for the Eddington approximate solution and for the exact solution. For $\tau > 1$ the difference is small, and both have the same limiting behaviour $T \to (3\tau/4)^{1/4}T_{\rm eff}$ as $\tau \to \infty$. Note this figure illustrates that $T_{\rm eff}$ is not directly related to any real temperature; the temperatures for both solutions change with τ, but $T_{\rm eff}$ is constant with τ.

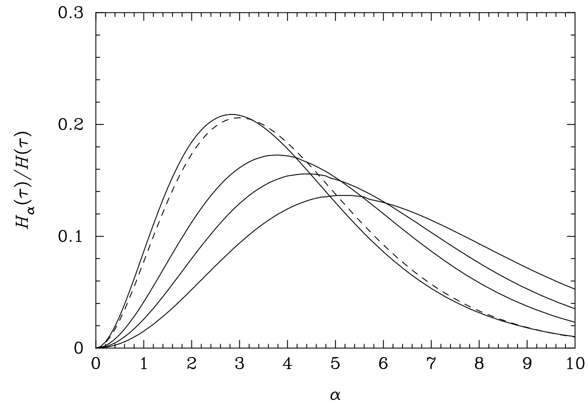

Figure 4.4: The value of $H_\alpha(\tau)/H$ for an LTE grey atmosphere at $\tau = 0$, 1, 2, and 4 (solid lines peaking from left to right) and value of $B_\alpha(T_{\rm eff})/H$ (dashed line).

Flux

The assumption of LTE gives us T as a function of τ, and so can easily calculate the monochromatic source function $S_\nu(\tau) = B_\nu(T(\tau))$. We can then calculate the monochromatic flux from the Milne equation

$$H_\nu(\tau) = \frac{1}{2}\int_\tau^\infty dt\, S_\nu(t)E_2(t-\tau) - \frac{1}{2}\int_0^\tau dt\, S_\nu(t)E_2(\tau - t). \tag{4.80}$$

In general, the flux will depend on the frequency, the effective temperature, and the optical depth. However, we can hide the dependency on the effective temperature as follows. In the Plank function, the temperature appears only in the combination $h\nu/kT$. Therefore we can simplify this equation by defining a frequency surrogate $\alpha \equiv h\nu/kT_{\rm eff}$ and by defining a flux surrogate $H_\alpha \equiv H_\nu d\nu/d\alpha$. If we further define $p(\tau) \equiv T_{\rm eff}/T(\tau)$, and use $F = \sigma T_{\rm eff}^4$, we obtain

$$\frac{H_\alpha(\tau)}{H} = \left(\frac{4\pi k^4}{h^3c^2\sigma}\right)\alpha^3\left[\int_\tau^\infty dt\,\frac{E_2(t-\tau)}{e^{\alpha p(t)}-1} - \int_0^\tau dt\, S_\nu(t)\frac{E_2(\tau-t)}{e^{\alpha p(t)}-1}.\right] \tag{4.81}$$

This expression depends on α and τ only. Figure 4.4 shows the shape of the flux F_α/F at $\tau = 0$, 1, 2, 4, and 8. The figure clearly shows the progressive reddening of the flux as it progresses to lower optical depth. Also marked in the figure is the flux from a black body emitting at the effective temperature; it is close to but different to the emergent flux. This demonstrates that even this simple atmosphere shows departures from a black body emitter.

Table 4.1: The $UBVRIJHKLKM$ and $uvby\beta$ Filter Systems

Filter	λ/μm	$\Delta\lambda/\lambda$
U	0.365	0.18
B	0.445	0.21
V	0.551	0.16
R	0.658	0.21
I	0.806	0.18
J	1.220	0.17
H	1.630	0.19
K	2.190	0.18
L	3.450	0.14
M	4.750	0.10
u	0.349	0.09
v	0.411	0.05
b	0.467	0.04
y	0.547	0.04
β_n	0.489	0.01
β_w	0.489	0.03

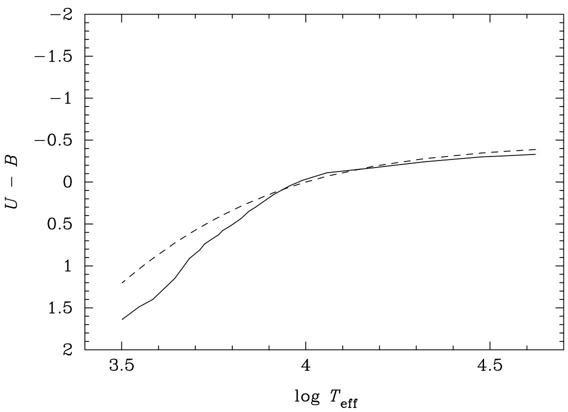

Figure 4.5: The observed relation between $T_{\rm eff}$ and $B-V$ for main sequence stars (solid line) and the relation predicted by an LTE grey atmosphere (dashed line).

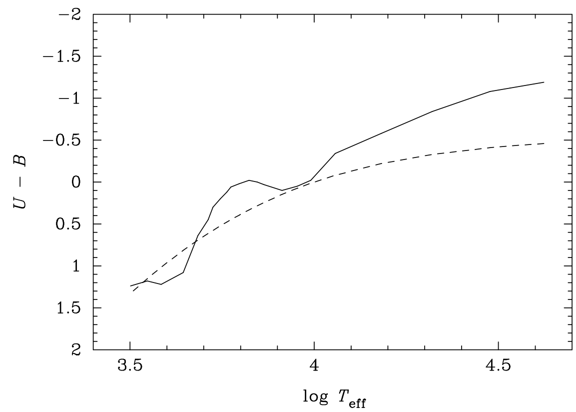

Figure 4.6: The observed relation between $T_{\rm eff}$ and $U-B$ for main sequence stars (solid line) and the relation predicted by an LTE grey atmosphere (dashed line).

Photometry and Colors

We measure the magnitude of a star by measuring the flux after passing its spectrum through a filter. The most common filters or are the $UBVRIJHKLM$ filters; UBV were Johnson's original filters; the rest were added later. These bands range from the atmospheric cut off at about 3200 Å into thermal infrared beyond 2.2 μm. The Strömgren $uvby\beta$ system is also used quite commonly in precision stellar photometry, as they are narrower (with band widths of less than 10% compared to around 20% for the Johnson filters), and so measure the spectrum of a star more finely. The mean wavelengths and bandwidths of these filters are given in table 4.1. Other systems include the Geneva, DDO, Vilnius, Thuan-Gunn, Hipparcos, Tycho, and SDSS systems.

The apparent magnitude m_X in a band X is defined as

$$m_X = -2.5 \log_{10} F_X + C_X \tag{4.82}$$

where F_X is the observed flux and C_X is a constant. In practice, magnitudes are measured by comparing the flux of a star to the flux of standard stars whose magnitudes are specified by the system. We often use X instead of m_X. The absolute magnitude M_X of a star in a band X is the apparent magnitude it would have if it were at a distance of 10 pc instead of its real distance d. From the inverse square law, we can see that

$$M_X = m_X - 5 \log_{10}(d/10 \text{ pc}). \tag{4.83}$$

We form colors be simply subtracting a pair of magnitudes (apparent or absolute; colors are independent of distance). Traditionally, the redder color is subtracted from the bluer color, e.g., $U-B$, $B-V$, and $V-R$ and A0V stars are defined to have zero color. Thus, if a star has a positive color, it is redder than an A0V star, and if it has a negative color, it is bluer than an A0V star.

The limits of the grey atmosphere become apparent when we compare F_ν to the observed fluxes from stars. Obviously, the flux from from a grey atmosphere shows no lines, but it also differs markedly from the shape of the continuum. Figure 4.5 shows the observed relation of $B-V$ color for main sequence stars and the relation predicted by an LTE grey atmosphere. The magnitudes were estimated by assuming that the response curves for B and V were top-hat function with central wavelengths and widths taken from Table 4.1. The zero-points were fixed by requiring that the color be zero for an effective temperature of 10,000 K, roughly appropriate for an A0V star. That is,

$$B - V = -2.5 \log_{10} \left(\frac{F_B(T_{\rm eff})}{F_V(T_{\rm eff})} \right) + 2.5 \log_{10} \left(\frac{F_B(10^4 \text{ K})}{F_V(10^4 \text{ K})} \right). \tag{4.84}$$

The colors are relatively close above $log_{10} T_{\rm eff} \approx 3.85$ or $T_{\rm eff} \approx 7000$, which corresponds to stars earlier than F2. However, for the cooler stars, the observed colors are significantly redder (more positive) than those predicted. Overall, though, the agreement is surprisingly good. However, Figure 4.6 shows the observed and predicted relations of $U-B$, and it tells a different story. There are large discrepancies, both in the shape of the curve and in the values of the color. The root of the problem is that U lies largely to the blue of the Balmer jump, the absorption feature caused by photoionization from the $n = 2$ level in neutral hydrogen, whereas B lies to the red. In A and F stars in particular, the Balmer jump is very strong, for reasons we will discuss later, and this causes the atmosphere to be very non-grey.

The conclusions we can draw from these comparisons is that the grey atmosphere is a useful didactic tool, and shows some of the qualitative behaviours of real stellar atmospheres. However, it does not show all of the qualititive behaviours – it has nothing like the Balmer jump – and is of very limited quantitative value.

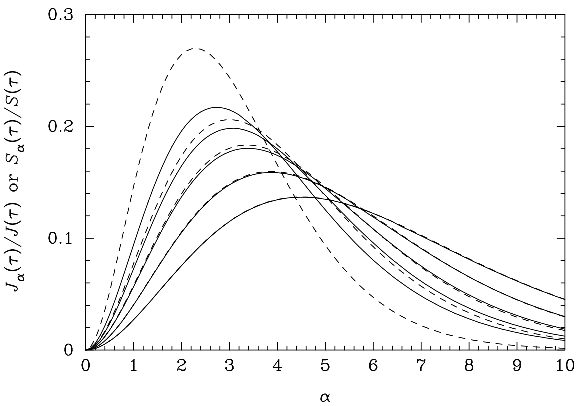

Figure 4.7: The value of $J_\alpha(\tau)/J$ (solid lines) and the value of $S_\alpha(\tau)/S(\tau)$ (dashed lines) for an LTE grey atmosphere at $\tau = 0$, 1, 2, 4, and 8 (peaking from left to right in each case).

Heating and Cooling

Figure 4.7 compares the mean intensity J_τ/J and the source function $S_\nu/S = B_\nu/B$ for $\tau = 0$, 1, 2, 4, and 8. It illustrates two important points. First, even though in the grey atmosphere the condition of radiative equilibrium requires that $J = S$, in general we do not have $J_\nu = S_\nu$. The departures are largest at small optical depths. However, at large optical depths we begin to approach the diffusion approximation for which $J_\nu \approx S_\nu$.

Second, we can identify which parts of J_ν are responsible for heating and cooling. J_ν will be result in heating (i.e., will deposit more energy than is emitted in this frequency range) is it is larger than S_ν. Similarly, J_ν will result in cooling (i.e., will deposit less energy than is emitted in this frequency range) if it is smaller than S_ν. For the grey atmosphere, the heating is provide by the Wien (high frequency) part of J_ν and the cooling by the Rayleigh-Jeans (low frequency) part. We can understand this from the dependence of the Planck curve on temperature. In the Wien regime we have

$$B_\nu(T) \approx \frac{2h\nu^3}{c^2} e^{-h\nu/kT}, \tag{4.85}$$

which drops very rapidly with T. In the Rayleigh-Jeans regime we have

$$B_\nu(T) \approx \frac{2k\nu^2}{c^2} T, \tag{4.86}$$

which obviously drops only linearly with T. Thus, as the temperature increases, the Wien region increases much more rapidly than the Rayleigh-Jeans region. The mean intensity at the surface arises from a range of optical depths, each of which has a source function that corresponds to a higher temperature and relatively larger in the Wien region than the source function. This is reflected in the mean intensity at the surface being relatively stronger in the Wien region than in the Rayleigh-Jeans region. Since radiative equilibrium requires that the integrated J and S be equal, the augmented Wien region must be compensated by a diminished Rayleigh-Jeans region. This is a simple example of a universal phenomena, that in general we have $S_\nu \neq J_\nu$, and so there are parts of the mean intensity that are responsible for heating and and other parts that are responsible for cooling.