RECENT PROGRESS IN HIGH PRECISION QED OF LIGHT HYDROGENLIKE ATOMS

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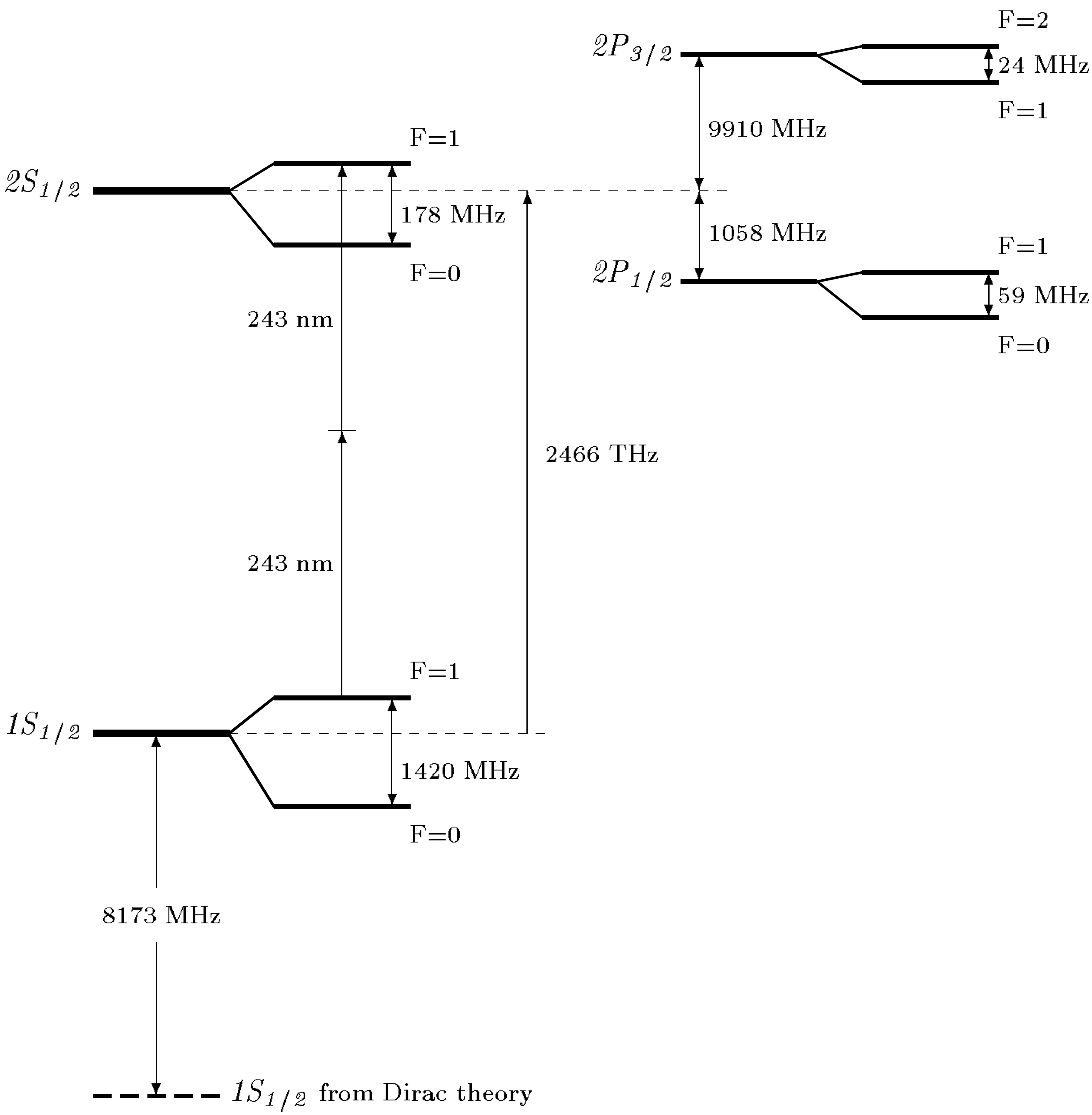
Abstract

The present status and recent developments in the high precision QED of light hydrogenic atoms are reviewed. We concentrate on Lamb shift in hydrogen and muonic hydrogen, and hyperfine splitting in muonium and hydrogen. Phenomenological implications for precise determination of the Rydberg constant, electron-muon mass ratio, proton charge radius, and other fundamental physical constants also will be discussed.

Outline

Energy Levels of the Hydrogen Atom



- Nonrelativistic QM does not explain Lamb shift
- Field Theory: Quantum electrodynamics

Theoretical Methods

Schrödinger equation in the Coulomb center with charge Z

$$(-\frac{\Delta}{2m}-\frac{Z\alpha}{r})\psi(\mathbf{r})=E_n\psi(\mathbf{r})$$

$$E_n=-\frac{m(Z\alpha)^2}{2n^2},\qquad n=1,2,3\ldots,$$

$$\mathbf{v}\sim\mathbf{Z}\alpha$$

Classical relativistic corrections

$$\sqrt{\mathbf{p}^2+m^2}\approx m+\frac{\mathbf{p}^2}{2m}-\frac{\mathbf{p}^4}{8m^3}+\ldots$$

Expansion over $\frac{\mathbf{p}^2}{m^2}\sim\mathbf{v}^2\sim(Z\alpha)^2$

Relativistic corrections are properly described by the Dirac energy levels in the field of an infinitely heavy Coulomb center

$$\mathbf{E_{nj}}=\mathbf{mf(n,j)},$$

$$
\mathbf{f(n,j)} = \left[1 + \frac{(\mathbf{Z}\alpha)^2}{\left(\sqrt{(\mathbf{j}+\frac{1}{2})^2 - (\mathbf{Z}\alpha)^2} + \mathbf{n} - \mathbf{j} - \frac{1}{2}\right)^2}\right]^{-\frac{1}{2}}
$$

$$
\approx 1 - \frac{(\mathbf{Z}\alpha)^2}{2\mathbf{n}^2} - \frac{(\mathbf{Z}\alpha)^4}{2\mathbf{n}^3}\left(\frac{1}{\mathbf{j}+\frac{1}{2}} - \frac{3}{4\mathbf{n}}\right)
$$

$$
-\frac{(\mathbf{Z}\alpha)^6}{8\mathbf{n}^3}\left[\frac{1}{(\mathbf{j}+\frac{1}{2})^3} + \frac{3}{\mathbf{n}(\mathbf{j}+\frac{1}{2})^2} + \frac{5}{2\mathbf{n}^3} - \frac{6}{\mathbf{n}^2(\mathbf{j}+\frac{1}{2})}\right] + \ldots
$$

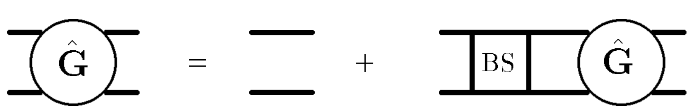
Small parameters: α/π, $Z\alpha$, m/M

Energy levels with account for leading recoil corrections

$$
E^{tot}_{njl} = (m+M) + m_r[f(n,j)-1] - \frac{m_r^2}{2(m+M)}[f(n,j)-1]^2 + L_{njl}
$$

High Order Corrections

Method of calculations: Bethe-Salpeter equation, nonrelativistic QED, etc.



- Binding (relativistic) corrections: expansion over $Z\alpha$

- Radiative (quantum electrodynamic) corrections: combined expansion over α/π and $Z\alpha$

The same physics corresponds to the corrections of different order in α/π at fixed order of $Z\alpha$

- Recoil corrections: expansion over m/M and $Z\alpha$

- Radiative-recoil corrections: expansion over m/M, $Z\alpha$, and α

- Nonelectromagnetic corrections: effects of strong and weak interactions (proton radius and structure, heavy particle vacuum polarization, electroweak boson exchanges, etc.)

Experimental Data

- Electron $g-2$ and Fine Structure Constant

  G. Gabrielse et al, 2006:

$$\frac{g}{2} = 1.001\ 159\ 652\ 180\ 85\ (76), \qquad \delta = 0.76 \times 10^{-12}$$

  Fine Structure Constant:

$$\alpha^{-1} = 137.035\ 999\ 710\ (90)\ (33)$$

The first error is due to experimental uncertainty, the second is due to $g-2$ theory

$$\alpha^{-1} = 137.035\ 999\ 710\ (96), \qquad \delta = 0.7 \times 10^{-9}$$

- Hydrogen: **Theodor W. Hänsch, Nobel Prize 2005**

  Niering et al, 2000:

$$f_{1S-2S} = 2\ 466\ 061\ 413\ 187.103\ (46)\ \text{kHz}, \quad \delta = 1.8 \cdot 10^{-14}$$

Schwob et al, 1999:

$$f_{2S_{\frac{1}{2}}-8D_{\frac{5}{2}}} = 770\ 649\ 561\ 581.1\ (5.9)\ \text{kHz}, \quad \delta = 7.7 \cdot 10^{-12}$$

Relative uncertainty of the $1S-2S$ frequency in hydrogen was reduced during the last decade by more than three orders of magnitude!

- Muonium

Mariam et al, 1982:

$$\Delta E_{HFS}(Mu) = 4\ 463\ 302.88\ (16)\ \text{kHz}, \quad \delta = 3.6 \cdot 10^{-8}$$

Liu et al, 1999:

$$\Delta E_{HFS}(Mu) = 4\ 463\ 302.776\ (51)\ \text{kHz}, \quad \delta = 1.1 \cdot 10^{-8}$$

Relative uncertainty of the hyperfine splitting in muonium was reduced by the factor 3

Theory in 2001

- **"Corrections of order $\alpha^2(Z\alpha)^6$ are the largest uncalculated contributions to the energy levels for S-states. The correction of this order is a polynomial in $\ln(Z\alpha)^{-2}$, starting with the logarithm cubed term. Both the logarithm cubed term and the contribution of the logarithm squared terms to the difference $\Delta E_L(1S) - 8\Delta E_L(2S)$ are known. However, the calculation of the respective contributions to the individual energy levels is still missing."**

- *"It is reasonable to take one half of the logarithm cubed term (which has roughly the same magnitude as the logarithm squared contribution to the interval $\Delta E_L(1S) - 8\Delta E_L(2S)$) as an estimate of the scale of all yet uncalculated logarithm squared contributions."*

- *"Uncertainties induced by the uncalculated contributions of order $\alpha^2(Z\alpha)^6$ constitute 14 kHz and 2 kHz for the $1S$- and $2S$-states, respectively."*

Eides, Grotch, Shelyuto, Phys. Rep. **342** *(2001)*

Corrections of Order $\alpha(Z\alpha)^n m$

$$\Delta E = G_{SE}(Z\alpha)\,\frac{\alpha(Z\alpha)^6}{\pi n^3}\left(\frac{m_r}{m}\right)^3 m,$$

where $G_{SE}(Z\alpha) = A_{60} + \left(A_{71}\ln[\frac{m(Z\alpha)^{-2}}{m_r}] + A_{70}\right)Z\alpha + \ldots.$

"Relativistic Bethe logarithms" A_{60} for many energy levels were calculated with high precision: *Jentschura, Pachucki (1996), Jentschura, Soff, Mohr (1997), Jentschura, Le Bigot, Mohr et al (2003), Le Bigot, Jentschura, Mohr et al (2003), Czarnecki, Jentschura, Pachucki (2005)*

A_{71} is known for a long time, *Karshenboim (1994)*:

$$A_{71} = \left(\frac{139}{64} - \ln 2\right)\pi$$

A_{70} and higher order terms never calculated perturbatively in $Z\alpha$.

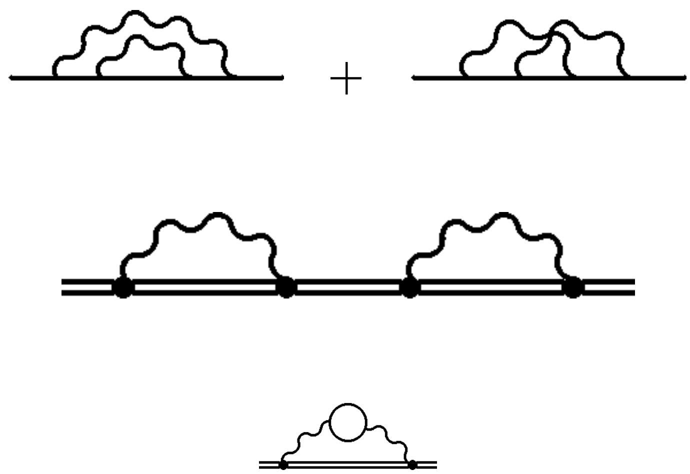
Spectacular Breakthrough:

Numerical calculation of the function $G_{SE}(Z\alpha)$ for low $Z = 1 - 5$: *Jentschura, Mohr, Soff (1999,2001), Jentschura, Mohr (2004,2005).*

The function $G_{SE}(Z\alpha)$ is now known with the numerical uncertainty 0.8 Hz for the $1S$-level in hydrogen, and 1.0 Hz for the $2P$-level in hydrogen. From the practical point of view these results completely solve all problems with calculation of the higher order corrections in $Z\alpha$ of the form $\alpha(Z\alpha)^n m$ in the foreseable future.

◇ **Important lesson:** *Difference between perturbative contribution with account for A_{60} and A_{71}, and numerical result with $G_{SE}(Z\alpha)$ is 13 kHz for $1S$ state in hydrogen! High order terms are large! It would be interesting to calculate directly nonlogarithmic term $\alpha(Z\alpha)^7$ an logarithmic contribution of order $\alpha(Z\alpha)^8$.*

Corrections of Order $\alpha^2(Z\alpha)^n m$



- All diagrams above and many others generate contributions of order $\alpha^2(Z\alpha)^6 m$

$$\Delta E = \Big[B_{63}\ln^3(Z\alpha)^{-2} + B_{62}\ln^2(Z\alpha)^{-2}$$

$$+B_{61}\ln(Z\alpha)^{-2} + B_{60}\Big]\,\frac{\alpha^2(Z\alpha)^6}{\pi^2 n^3}\,m$$

- Coefficients B_{62}, B_{61}, B_{60} were calculated: *Pachucki (2001), Jentschura, Nandori (2002), Pachucki, Jentschura (2003), Jentschura (2003,2004)*

- *Another surprise: Contribution of single-logarithm dominates!*

$$\Delta E(1S) = \Big[-0.296\ln^3(Z\alpha)^{-2} - 0.640\ln^2(Z\alpha)^{-2}$$

$$+49.803\ln(Z\alpha)^{-2} - 61.6\Big]\frac{\alpha^2(Z\alpha)^6}{\pi^2}m$$

$$\approx [-282 - 62 + 490 - 61]\frac{\alpha^2(Z\alpha)^6}{\pi^2}m,$$

$$[-28.38 - 6.23 + 49.26 - 6.2]\ \mathrm{kHz} = 8.45\ \mathrm{kHz}$$

• Red Flag: *Prediction that an error is about 14 kHz marginally survived but for a wrong reason!*

Estimate of uncalculated terms is always an art, not science!

◇ **Still not the end of the story!** *Numerical error of calculation of* B_{60} *Pachucki, Jentschura (2003), Jentschura (2004) is about 15%. It translates into* 0.9 *kHz for the* $1S$*-state and* 0.1 *kHz for the* $2S$*-state.*

◇ **What about even higher order corrections of order** $\alpha^2(Z\alpha)^7$**, etc.?**

• Purely numerical calculation *Yerokhin, Indelicato, Shabaev (2005)* without expansion in $Z\alpha$ produced for $1S$ level in hydrogen

$$B_{60} + \textit{higher orders in } \mathrm{Z}\alpha = -127 \pm 30\%$$

to be compared with

$$B_{60} = -61.6\ (3) \pm 15\%$$

◇**Ground state energy in hydrogen shifts by 7 kHz!**

◇But *Yerokhin, Indelicato, Shabaev (2005)*: extrapolation of the numerical result to $Z\alpha = 0$ does not reproduce semianalytic B_{60}.

◇**More work is needed both on numerical side and with expansion in $Z\alpha$!**

◇ **Meanwhile we split the difference and assume that the theoretical error of these corrections is about 4 kHz for $1S$ state in hydrogen (pure speculation!)**

Normalized Lamb Shift Difference

$$\Delta_n = n^3 L(nS) - L(1S)$$

- All terms proportional $1/n^3$ cancel in Δ_n

◇ *Circa 2001:*

$$\Delta_2 = 187\ 231\ (5)\ \textbf{kHz}$$

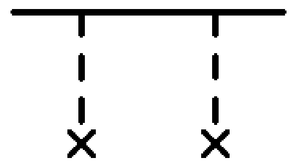
◇ *Today (Czarnecki, Jentschura, Pachucki (2005)):*

$$\Delta_2 = 187\ 225.70\ (5)\ \textbf{kHz}$$

- **All other Δ_n are also calculated with uncertainty about 0.1 kHz.**

◇ **When experimental data will become available this Δ_n will allow to reduce relative accuracy of the Rydberg constant from 6.6×10^{-12} to about 10^{-14}!**

Corrections of Order $\alpha^3(Z\alpha)^5 m$



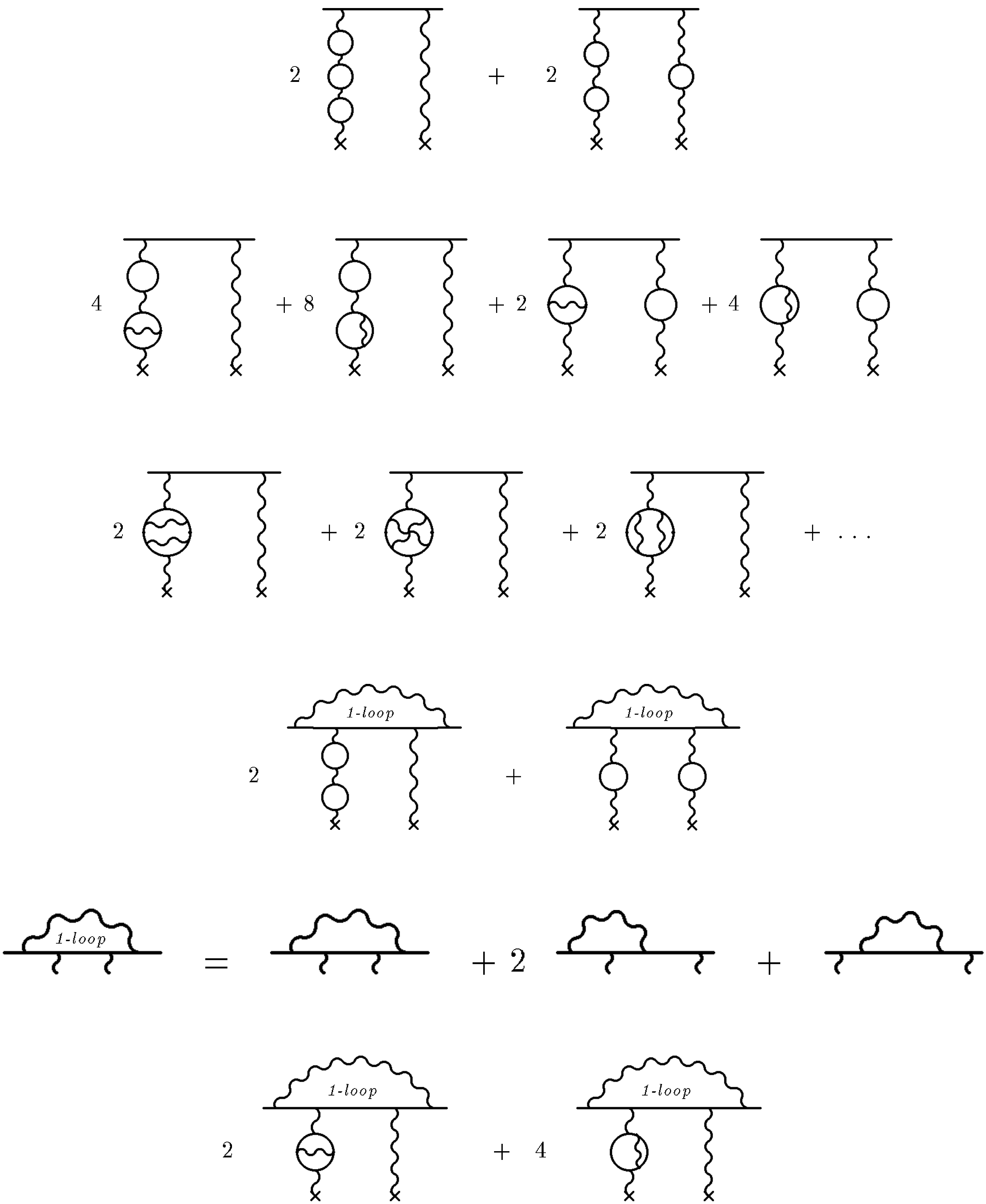
Corrections of order $\alpha^3(Z\alpha)^5$ are generated by three-loop radiative insertions in the skeleton diagram with two external photons.

They originate from small distances (large momenta) and may be calculated in the scattering approximation.

Corrections of order $\alpha^3(Z\alpha)^5 m$ are suppressed by the factor α/π in comparison with $\alpha^2(Z\alpha)^5 m$, and may be about 1 kHz for the $1S$-state and about 0.1 kHz for the $2S$-state.

We expect that numerically dominant part of the corrections of order $\alpha^3(Z\alpha)^5$ will be generated by the gauge invariant set of diagrams with insertions of three radiative photons in the electron line in the skeleton diagrams.

- Current status. All corrections connected with the diagrams containing at least one one-loop or two-loop polarization insertion, and some other diagrams were obtained *Eides, Shelyuto (2003,2006)*


2 + 2
4 + 8 + 2 + 4
2 + 2 + 2 + . . .
1-loop
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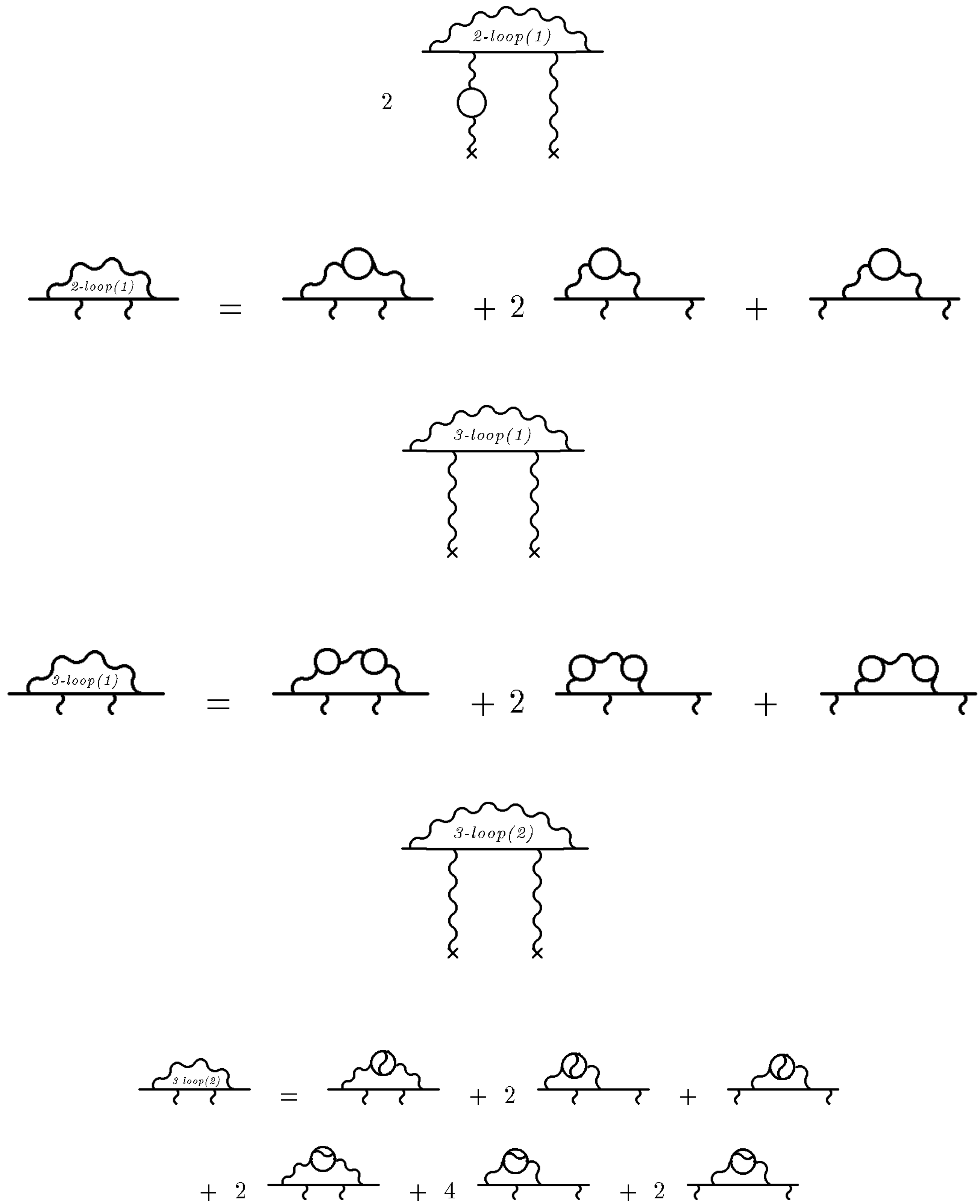
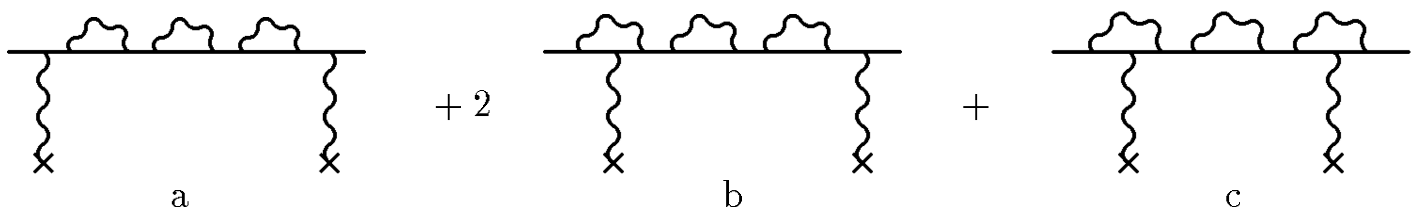
2 + 4



Contribution of all these diagrams

$$\Delta E = 2.651\ 9\ (6)\frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3}\left(\frac{m_r}{m}\right)^3 m\,\delta_{l0}.$$

Reducible insertions in the electron line



$$\Delta E = -5.\ 321\ 93\ (1)\ \frac{\alpha^3(Z\alpha)^5}{\pi^2 n^3}\left(\frac{m_r}{m}\right)^3\ m\,\delta_{l0}.$$

◇ **Work on calculation of the remaining contributions of order $\alpha^3(Z\alpha)^5$ is in progress now.**

Recoil Corrections

- Only the leading logarithm squared contribution to the recoil correction of order $(Z\alpha)^7(m/M)$ is known now *Pachucki, Karshenboim (1999), Melnikov, Yelkhovsky (1999)*. Numerically this contribution is below 1 kHz. Due to linear dependence of the recoil correction on the electron-nucleus mass ratio, the respective contribution to the hydrogen-deuterium isotope shift.

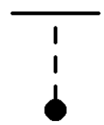
- Numerical results for recoil corrections without expansion over $Z\alpha$ *Shabaev, Artemeyv, Beier et al (1998), Shabaev (2000)* indicate that contribution of the single logarithmic and nonlogarithmic recoil corrections of order $(Z\alpha)^7(m/M)$ is about $0.64(1)$ kHz for the $1S$ level in hydrogen, and exceeds total logarithm squared contribution. It would be interesting to calculate respective coefficients perturbatively.

Radiative-Recoil Corrections

- Only the leading logarithm squared radiative-recoil contribution of order $\alpha(Z\alpha)^6(m/M)$ is known now *Pachucki, Karshenboim (1999), Melnikov, Yelkhovsky (1999)*. This contribution constitutes 1.52 kHz and 0.19 kHz for the $1S$ and $2S$ levels in hydrogen, respectively. Experience with the large contributions generated by the nonleading terms of order $\alpha^2(Z\alpha)^6(m/M)m$ and of order $(Z\alpha)^7(m/M)m$ shows that the nonleading terms could be large and should be calculated.

Nuclear Size and Structure Corrections

Leading Nuclear Size Correction



$$\Delta E = \frac{2(Z\alpha)^4}{3n^3} m_r^3 \langle r^2 \rangle \delta_{l0}.$$

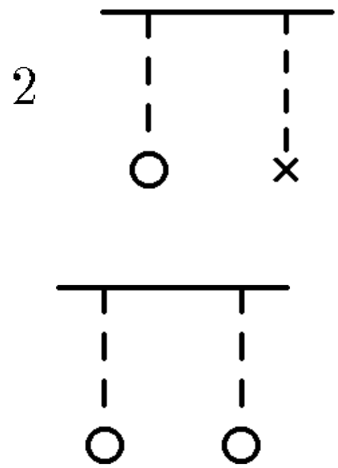
◇ **Main news is the new higher value of the proton radius**
Sick (2003) from world data on electron-proton scattering

$$r_p = 0.895\ (18)\textbf{fm}$$

Still leads to a huge uncertainty

$$\Delta E(1S) = 1\ 253\ (50)\ \textbf{kHz}$$

Nuclear Size Corrections of Order $(Z\alpha)^5 m$



$$\Delta E = -\frac{m(Z\alpha)^5}{3n^3} m_r^3 \langle r^3 \rangle_{(2)},$$

where the third Zemach moment is

$$\langle r^3 \rangle_{(2)} = \frac{48}{\pi} \int_0^\infty \frac{dk}{k^4} \left(G_E^2(-k^2) - 1 + \frac{\langle r^2 \rangle}{3} k^2 \right)$$

$$= \int d^3 r_1 d^3 r_2 \rho(r_1) \rho(r_2) |\mathbf{r}_1 - \mathbf{r}_2|^3$$

Model-independent result from analysis of the electron-proton scattering (*Friar, Sick (2005)*)

$$\langle r^3 \rangle_{(2)} = 2.71\ (13)\ \textbf{fm}^3$$

$\Delta E = -40.0\ (1.9)$ **Hz for the** $1S$ **state and** $\Delta E = -5.0\ (2)$ **Hz for the** $2S$ **state in hydrogen**

Main Sources of Theoretical Uncertainties

• Corrections $\alpha^2(Z\alpha)^6 m$ and $\alpha^2(Z\alpha)^7 m$. Discrepancy between calculation of B_{60} and numerical calculation of all corrections $\alpha^2(Z\alpha)^n m$. Estimate of uncertainty is about 4 kHz for $1S$ level in hydrogen.

◇ Uncalculated corrections of order $\alpha^3(Z\alpha)^5 m$. They are pure numbers and could be as large as 1 kHz for the $1S$-state and about 0.1 kHz for the $2S$-state.

◇ There are indications that logarithmic and nonlogarithmic recoil corrections of order $(Z\alpha)^7(m/M)$ are about $0.64(1)$ kHz. Direct calculation would be interesting.

• Single logarithmic and nonlogarithmic radiative-recoil contributions of order $\alpha(Z\alpha)^6(m/M)m$ may be estimated as one half of the leading logarithm squared contribution, this constitutes about 0.8 kHz and 0.1 kHz for the $1S$ and $2S$ levels in hydrogen. However, this could be too low. Calculation is needed.

◇ 6 kHz for the $1S$-state and 0.8 kHz for the $2S$-state are reasonable estimates of the total theoretical uncertainty of the expression for the Lamb shift.

◇ Agreement between theory and experiment is

satisfactory.

Classic $2S_{1/2} - 2P_{1/2}$ Lamb Shift

	ΔE (kHz)
Newton,Andrews, Unsworth (1979)	$1\ 057\ 862\ (20)$
Lundeen,Pipkin (1981)	$1\ 057\ 845\ (9)$
Palchikov,Sokolov, Yakovlev (1983)	$1\ 057\ 857.\ 6\ (2.1)$
Hagley,Pipkin (1994)	$1\ 057\ 839\ (12)$
Wijngaarden,Holuj, Drake (1998)	$1\ 057\ 852\ (15)$
Schwob,Jozefovski, de Beauvoir et al (1999)	$1\ 057\ 845\ (3)$
Theory, $r_p = 0.805\ (11)$ **fm**	$1\ 057\ 822.0\ (0.8)\ (3.5)$
Theory, $r_p = 0.862\ (12)$ **fm**	$1\ 057\ 840.6\ (0.8)\ (4.0)$
Theory, $r_p = 0.895\ (18)$ **fm**	$1\ 057\ 851.9\ (0.8)\ (6.3)$

$1S$ Lamb Shift

	ΔE (kHz)
Weitz,Huber,Schmidt-Kahler et al (1995)	$8\ 172\ 874\ (60)$
Berkeland,Hinds, Boshier (1995)	$8\ 172\ 827\ (51)$
Bourzeix,de Beauvoir, Nez et al (1996)	$8\ 172\ 798\ (46)$
Udem,Huber, Gross et al (1997)	$8\ 172\ 851\ (30)$
Schwob,Jozefovski, de Beauvoir et al (1999)	$8\ 172\ 837\ (22)$
Theory, $r_p = 0.805\ (11)$ **fm**	$8\ 172\ 664\ (6)\ (28)$
Theory, $r_p = 0.862\ (12)$ **fm**	$8\ 172\ 812\ (6)\ (32)$
Theory, $r_p = 0.895\ (18)$ **fm**	$8\ 172\ 902\ (6)\ (50)$

◇ **Goal: Reduce theoretical uncertainty for $1S$ below 1 kHz!**

◇ **But: Potential problem with the Lamb shift in He^+ ion: 2σ discrepancy with theory**

He^+ Lamb Shift

Theory

$$L_{th}(2S-2P, He^+) = 14\,041.46\ (3)\ \text{MHz}$$

Old experiment, van Wijngaarden, Kwela, Drake (1991):

$$L(2S-2P, He^+) = 14\,042.52\ (16)\ \text{MHz}$$

New experiment, Drake et al (2000):

$$L(2S-2P, He^+) = 14\,041.12\ (17)\ \text{MHz}$$

Lamb Shift and Proton Radius

$$\Delta E_L^{theor}(1S) = 8\ 172\ 902\ (6)\ (50)\ \text{kHz}$$

where $r_p = 0.895\ (18)$ fm. The first error is determined by the yet uncalculated contributions to the Lamb shift, and the second reflects the experimental uncertainty in the measurement of the proton rms charge radius.

The uncertainty of the experimentally measured proton charge radius currently determines the uncertainty of the theoretical prediction for the Lamb shift.

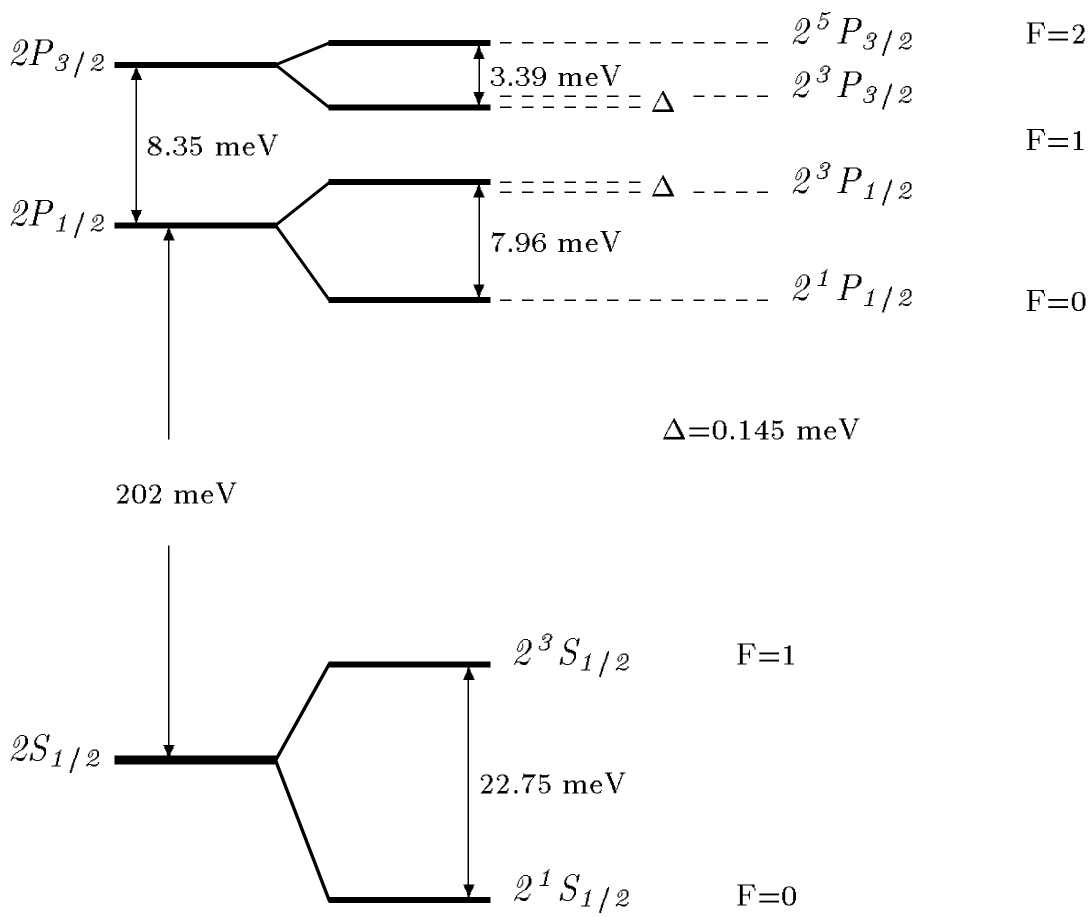
Invert the problem and obtain the "optical" proton charge radius

$$r_p = 0.875\ (6)\ fm$$

Atomic spectroscopy became a source of precise data on the nucleon properties

Muonic hydrogen is even better for measuring the proton charge radius!

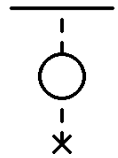
Lamb Shift in Muonic Hydrogen



Two main features:

1) Closed electron loops are enhanced,
2) Proton radius contribution is enhanced.

Both are due to $m_\mu/m_e \approx 200$.



- Closed Electron Loops

The Coulomb potential is distorted by polarization at distances $r_C \sim 1/m_e$.

Atomic radius in ordinary hydrogen: $r_B \sim 1/(m_e Z\alpha)$.

Atomic radius in muonic hydrogen: $r_B^\mu \sim 1/(m_\mu Z\alpha)$

$r_B/r_C \sim 1/(Z\alpha) \sim 137$, but $r_B^\mu/r_C \sim m_e/(m_\mu Z\alpha) \sim 0.7$!

Contribution of closed electron loops is enhanced because muon sees unscreened by polarization proton charge. Binding is stronger, S level shifts down.

The Lamb shift is due to competition of two effects:

a) Spreading of the orbiting lepton, which decreases binding anf shifts the S level up

b) Vacuum polarization, which screens the charge of the nucleus. The orbiting lepton sees the charge of the nucleus at a finite distance, and vacuum polarization shifts the S level down

In ordinary hydrogen lepton spreading wins, and the S level shifts up.

In muonic hydrogen vacuum polarization wins, and the S level shifts down. In muonic hydrogen $2S$-level is lower than $2P$.

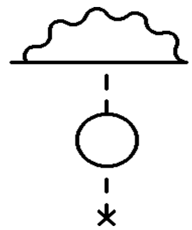
- The leading proton size contribution ($\sim m^3 r_p^2$) is strongly enhanced, it is the second largest contribution after the polarization correction

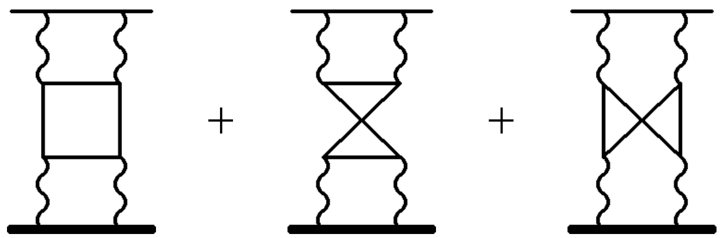
In ordinary hydrogen: $m_e^3 r_p^2 = (m_e r_p)^2 m_e$.

In muonic hydrogen: $m_\mu^3 r_p^2 = (m_\mu r_p)^2 m_\mu$

Relative enhancement: $(m_\mu/m_e)^2$!!!

Theoretical Problems with Muonic Hydrogen

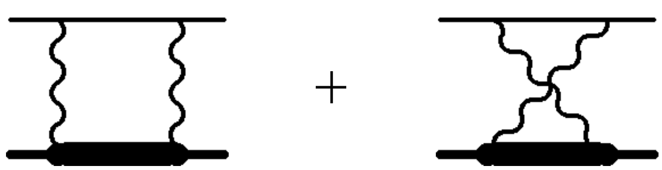




- Purely electrodynamic uncertainties are introduced by the uncalculated nonlogarithmic contribution of order $\alpha^2(Z\alpha)^4$ generated by the diagrams with radiative photon insertions in the graph for leading electron polarization.

- Uncalculated light by light contributions of order $\alpha^2(Z\alpha)^3$

Total electrodynamic uncertainty easily could be as large as 0.004 meV.



- Nuclear polarizability contribution of order $(Z\alpha)^5 m$ contributes uncertainty about 0.002 meV

• Dependence of the Lamb shift on the proton rms radius and the third Zemach moment

$$\Delta E(2P-2S) = -5.225\ \langle r_p^2 \rangle\ \text{meV}$$

• The third Zemach moment was determined model independently from the world data on electron-proton scattering *Friar, Sick (2005)*

$$\langle r^3 \rangle_{(2)} = 2.71\ (13)$$

Respective contribution to the Lamb shift is

$$\Delta E(2P-2S) = 0.0247\ (12)\ \text{meV}$$

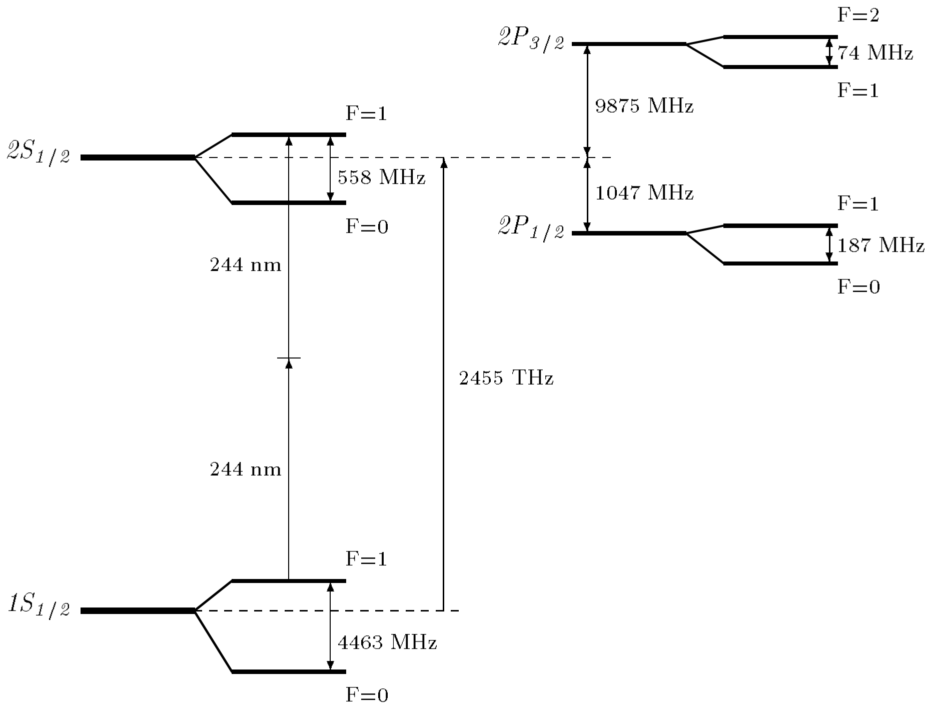
Current value of the Lamb shift with experimental proton rms radius and experimental third Zemach moment

$$\Delta E(2P-2S) = 206.07\ (18)\text{meV}$$

Uncertainty is determined by the proton radius!

- When the experiment achieves accuracy about 0.008 meV we determine the proton charge radius with relative accuracy about 0.1%. Say for the Sick value $r_p = 0.895$ fm, the error will decrease by the factor 20, from 0.018 fm to 0.0009 fm. Quite an improvement!

Hyperfine Splitting in Muonium and the Electron-Muon Mass Ratio



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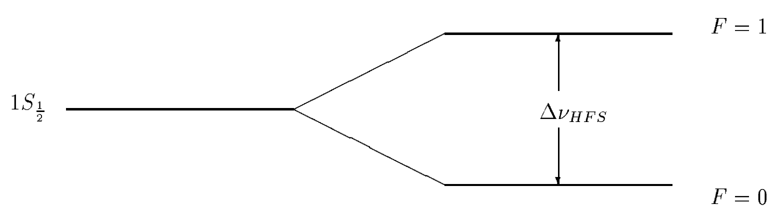


Fermi energy is proportional to the electron-muon mass ratio:

$$E_F = \frac{8}{3}(Z\alpha)^4(1+a_\mu)\frac{m}{M}\left(\frac{m_r}{m}\right)^3 mc^2$$

$$= \frac{16}{3} Z^4 \alpha^2 (1 + a_\mu) \frac{m}{M} \left(\frac{m_r}{m}\right)^3 ch\, R_\infty,$$

Numerically:

$$E_F = 4\ 459\ 031.936\ (518)\ (30)\ \text{kHz},$$

the uncertainty in the first brackets is due to the uncertainty of the best direct experimental value of muon-electron mass ratio $M/m = 206.768\ 277\ (24)$, and the uncertainty in the second brackets is due to the uncertainty of the fine structure constant α.

The dominant error is due to the electron-muon mass ratio!

Exact hyperfine splitting interval:

$$\Delta E_{HFS} = E_F (1 + \text{corrections})$$

Mass ratio from the experimental value of HFS and the most precise value of α

$$\frac{M}{m} = 206.768\ 282\ 9\ (23)\ (14)\ (32),$$

where the first error comes from the experimental error of the hyperfine splitting measurement, the second comes from the error in the value of the fine structure constant α, and the third from an estimate of the yet unknown theoretical contributions.

Combining all errors we obtain the mass ratio

$$\frac{M}{m} = 206.768\,282\,9\;(41) \qquad \delta = 2.0 \cdot 10^{-8},$$

which is almost six times more accurate than the best direct experimental value.

The largest contribution to the uncertainty of the indirect mass ratio is supplied by the unknown theoretical contributions to hyperfine splitting. This sets a clear task for the theory to reduce the contribution of the theoretical uncertainty in the error bars to the level below two other contributions to the error bars. It is sufficient to this end to calculate all contributions to HFS which are larger than 10 Hz. This would lead to further reduction of the uncertainty of the indirect value of the muon-electron mass ratio. There is thus a real incentive for improvement of the theory of HFS to account for all corrections to HFS of order 10 Hz, created by the recent experimental and theoretical achievements.

Another reason to improve the HFS theory is provided by the perspective of reducing the experimental uncertainty of hyperfine splitting below the weak interaction contribution. In such a case, muonium could become the first atom where a shift of atomic energy levels due to weak interaction would be observed.

Hyperfine Splitting in Hydrogen

Radiation with 21 cm wavelength (at 1420 MHz) permeats the Galaxy.

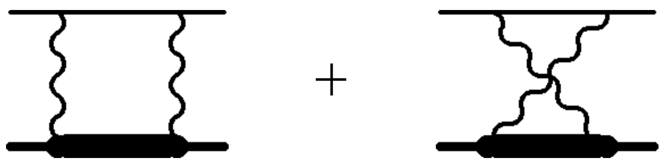
Hyperfine splitting in the ground state of hydrogen was measured precisely about thirty years ago (1971)

$$\Delta E_{HFS}(H) = 1\ 420\ 405.751\ 766\ 7\ (9)\ \mathrm{kHz} \qquad \delta = 6 \cdot 10^{-13}$$

For many years, this hydrogen maser measurement remained the most accurate experiment in modern physics. Only recently the accuracy of the Doppler-free two-photon spectroscopy achieved comparable precision.

The sum of all nonrecoil corrections to hyperfine splitting:

$\Delta E_{HFS}(H) = 1\ 420\ 452.04\ (2)$ kHz, where the error comes from the experimental error of the proton anomalous magnetic moment κ. Theoretical error is about 1 Hz! It was reduced by three orders of magnitude since 1988.



Diagrams for nuclear polarizability correction of order $(Z\alpha)E_F$

Until recently the stumbling block on the road to a more precise theory of hydrogen hyperfine splitting was the inability to calculate the polarizability contribution. It was first calculated by Faustov and Martynenko (2002) and improved by Nazaryan et al (2006).

The theoretical uncertainty of the total recoil contribution to hydrogen hyperfine splitting was reduced to 0.8 kHz. The current theoretical result for hydrogen hyperfine splitting is

$$\Delta E_{HFS}(H)_{th} = 1\ 420\ 403.1\ (8)\ \text{kHz}.$$

There is a discrepancy between theory and experiment which is more than 3σ!

Hyperfine Splitting in Deuterium

Hyperfine splitting in the ground state of deuterium was measured precisely about thirty years ago (1971)

$\Delta E_{HFS}(D) = 327\,384.352\,521\,9\ (17)\ \mathrm{kHz} \qquad \delta = 5.2 \cdot 10^{-12}$

The sum of all nonrecoil corrections to hyperfine splitting:

$\Delta E_{nrec}(D) = 327\,339.147\ (4)$ kHz,

where the error comes from the experimental error of the deuteron anomalous magnetic moment.

For many years nobody could explain the difference

$$\Delta E^{exp}_{HFS}(D) - \Delta E_{nrec}(D) = 45.2 \text{ kHz}. \qquad (1)$$

A breakthrough was achieved by Khriplovich, Milshtein, and Petrosyan (1996) who analytically calculated the deuterium recoil, structure and polarizability corrections in the zero range approximation. The analytic result for the difference is numerically equal 44 kHz This calculation was later improved in a number of papers.

There are topics in the field not covered in this talk. For more details on all these and many other topics see

- M. I. Eides, H. Grotch, and V. A. Shelyuto, Theory of Light Hydrogenic Bound States, Springer 2007