THE BAYESIAN CRAMÉR-RAO LOWER BOUND IN ASTROMETRY

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RESUMEN

La determinación de la máxima precisión que es posible obtener en la medición de la ubicación de un objeto puntual ha sido un tópico de interés permanente dentro de la comunidad astrométrica. El así denominado límite (no-paramétrico o no-Bayesiano) de Cramér-Rao (CR a partir de ahora) nos proporciona un límite inferior para la varianza con la cual es posible estimar la posición de un objeto puntual. Esto ha sido recientemente estudiado por Mendez et al. (2013, 2014, 2015). En este trabajo presentamos una aproximación distinta al mismo problema (Echeverria et al. 2016), haciendo uso de un esquema Bayesiano del límite de CR, el cual presenta varias ventajas con respecto al escenario paramétrico.

ABSTRACT

A determination of the highest precision that can be achieved in the measurement of the location of a stellar-like object has been a topic of permanent interest by the astrometric community. The so-called (parametric, or non-Bayesian) Cramér-Rao (CR hereafter) bound provides a lower bound for the variance with which one could estimate the position of a point source. This has been studied recently by Mendez *et al.* (2013, 2014, 2015). In this work we present a different approach to the same problem (Echeverria *et al.* 2016), using a Bayesian CR setting which has a number of advantages over the parametric scenario.

Key Words: astrometry — methods: analytical — methods: data analysis — methods: statistical

1. INTRODUCTION

Astrometry is the foundation of classical astronomy and modern astrophysics, and it remains a cornerstone of the field for the 21$^{\mathrm{st}}$ century. Nowadays, astronomers take for granted resources such as the ESA Hipparcos mission, which yielded a catalogue of more than 100,000 stellar positions to a precision of 1 milli-arcsecond, and look forward to the results of the ESA Gaia astrometric satellite which will deliver a catalogue of over 10^9 stars, with precisions better than 10-20 micro-arcseconds for objects brighter than $V = 15$ mag.

A determination of the highest precision that can be achieved in the measurement of the location of a stellar-like object in the sky, as mapped by a detector, has been a topic of permanent interest by the astrometric community. One of the tools used to characterize this maximum precision is the so-called (parametric, or non-Bayesian) Cramér-Rao (CR hereafter) bound, which provides a lower bound for the variance with which one could estimate - using *any* unbiased estimator - the position of a point source. In astrometry the CR bound offers meaningful closed-form expressions that can be used to analyze the complexity of the inference task in terms of key observational and design parameters such as: source properties (flux, shape), position of the object in the measuring array (usually a CCD), pixel resolution of the instrument and other design specifications of the detector, the structure of the background, and so on. Mendez *et al.* (2013, 2014, 2015) have developed closed-form expressions for this bound and have studied its structure and dependency with respect to important observational parameters. Complementing these results, Lobos *et al.* (2015) have studied the conditions under which the CR bound can be achieved by a practical estimator.

In this contribution, we present a summary of a novel analysis (Echeverria *et al.* 2016) of the best precision that can be achieved to determine the location of a point source on a CCD-like detector array in a Bayesian CR setting (BCR hereafter). In this work we extend our previous analysis, transiting from a classical parametric setting (where the position is considered fixed, but unknown), to a richer Bayesian setting, where the position becomes an unknown random object but we have access to prior knowledge in terms of the distribution of the object position. This changes in a fundamental way the nature of the inference problem: from a parametric context - in which we are estimating a constant parameter from a set of random observations - to a

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setting in which we estimate a random object (that is to say, the position is modeled as a random variable) from observations that are statistically dependent with the position.

2. SCOPE & HIGHLIGHTS OF THE BAYESIAN SCENARIO

A key new element of the Bayesian setting is the introduction of a prior distribution of the object position: In our work we quantify and analyze in a systematic way how much is the gain in astrometric performance from the use of a prior distribution of the object position, not available in the classical parametric setting. We also derive new closed-form expressions for the BCR as well as expressions to estimate the gain in astrometric precision. An insightful corollary of this analysis is that the Bayes setting *always* offers a better performance (tighter bound) than the parametric setting, even in the worse-case prior (i.e., that of a uniform distribution or of a very loose prior). This is a very interesting and important result, because it prompts efforts to approach that bound (see next paragraph, and our full paper).

We have also evaluated numerically the gain of the Bayes setting with respect to the parametric scenario under realistic experimental conditions: We find that the gain in performance is significant for various observational regimes, particularly in the case of faint objects, or when the observations are taken under poor conditions. We also demonstrate that the performance gains disclosed in our theoretical analysis can be achieved with the minimum mean-square-error estimator, which has a practical implementation (the conditional expectation).

Finally, we present an example of what could be achieved using the Bayesian approach in terms of the astrometric precision of positional measurements with new observations of varying quality, when we incorporate as prior information data from existing (real) catalogues. To this end, we have used the USNO catalogue near the Southern Galactic Pole (SGP) to evaluate the number of objects that will pass a certain positional uncertainty cut (in this case of 0.1 arcsec) using the classical CR bound in comparison with those that will satisfy the same positional restriction using the BCR, when new observations with a 1 m telescope, under different seeing conditions are performed, and the catalogue positions from the USNO are used as priors.

In the histograms of Figure 1, light grey represents the number of objects that will pass the positional selection criteria given the catalogue uncertainties directly, dark grey those that are added by new observations on a typical 1 m telescope+CCD not using any a priori knowledge. Finally, the black histogram represents the addition of objects from a Bayesian treatment, using the USNO declared uncertainties and positions as prior.

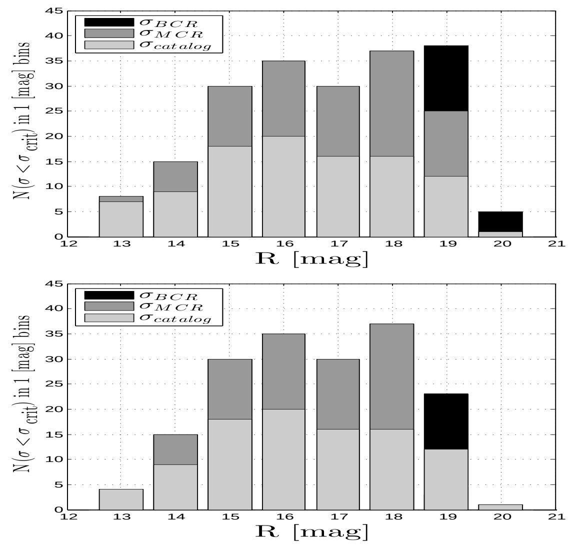

Fig. 1. Histogram of objects toward the SGP from USNO (see text for details) as a function of R-band magnitude.

As we can see, and as we would expect, the Bayesian setting has its largest impact on low SNR objects. On the upper panel, at 19th magnitude we have 38 objects in the Bayesian scenario, and 25 objects in the parametric approach, i.e., an increase. of more than 50%. In the right panel the increase is even more dramatic, 23 objects in the Bayesian approach, and 12 from the catalogue (equal to the number from CR), an increase of about 90%.

This shows the potential usefulness of a Bayesian approach to “rescue” objects at very low SNR, where the use of prior information becomes very relevant. On the other hand, at high SNR, the new observations are sufficiently informative so that the Bayesian approach becomes less relevant.

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