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SOFTENED POTENTIALS AND THE MULTIPOLAR EXPANSION

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RESUMEN

Cuando se calcula el desarrollo multipolar del potencial gravitatorio, los distintos multipolos quedan bien definidos, correspondiendo cada uno a una suma finita de términos de la serie. Sin embargo, al usar el potencial gravitatorio en simulaciones numéricas, suele desarrollarse en serie una versión *suavizada* del mismo. Ocurre que, en estos casos, el desarrollo multipolar estándar que suele utilizarse ya no aísla los multipolos, sino que cada uno de ellos queda distribuido en infinitos términos. En este artículo se muestra cómo recuperar los multipolos completos en estos casos. Afortunadamente, la diferencia entre usar multipolos incompletos y completos es despreciable en los casos de interés, por ejemplo, en su uso en códigos árbol.

ABSTRACT

When the gravitational potential is developed in a multipolar series, each multipole is well defined and corresponds to a finite sum of terms in the series. In order to use the gravitational potential in numerical simulations, however, a multipolar expansion is usually applied to a *softened* Newtonian potential. It turns out that the commonly used multipolar expansion in this case no longer isolates each multipole as in the former case; instead, each multipole is spilled over an infinity of terms. In this paper we show how to recover the complete multipoles. Fortunately, the overall effect of using incomplete multipoles instead of complete ones turns out to be negligible in the cases of interest, for example, in its use in tree codes.

Key Words: **GALAXY: KINEMATICS AND DYNAMICS — METHODS: *N*–BODY SIMULATIONS — METHODS: NUMERICAL**

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The multipolar expansion of the gravitational Newtonian potential, which is the result of a series development of the inverse of the distance between a source point and a field point (e.g., Kellogg 1954, Chap. V), is a widely used tool in N-body simulations of stellar systems (van Albada 1982; Villumsen 1982; White 1983; Aguilar & White 1985), becoming widespread since the bursting of tree-codes (Barnes & Hut 1986, 1989).

When an N-body distribution is to be advanced in time in order to simulate its gravitational evolution, it is customary to soften the potential. There is a numerical motivation for this softening, namely, to avoid the pole generated when two particles are too close. But there are also dynamical motivations, although different authors give different ones, and different problems may require a softening for different reasons. A particularly clear account of this state of affairs may be found in Dehnen(2001), § 1.1.

Following, e.g., White (1983), the standard multipolar expansion of a softened Newtonian potential at position $\mathbf{r}$ generated by a point mass m located at position $\mathbf{r}'$, is given by

$$-\frac{Gm}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+\varepsilon^2}} = -Gm\left\{\frac{1}{p}+\frac{1}{p^3}\mathbf{r}\cdot\mathbf{r}'+ \frac{1}{p^5}\left[\frac{3}{2}(\mathbf{r}\cdot\mathbf{r}')^2-\frac{1}{2}r_<^2p^2\right]+\cdots\right\}, \quad (1)$$

where G is the gravitational constant, ε is the softening of the potential, $r_< = \min(|\mathbf{r}|, |\mathbf{r}'|)$, and $p = \sqrt{r_>^2 + \varepsilon^2}$, where $r_> = \max(|\mathbf{r}|, |\mathbf{r}'|)$. Borrowing from the nomenclature of non-softened potentials, the first term is called the monopole, the second one the dipole, the third one the quadrupole, and so on. The foregoing expression may be rewritten in terms of the Legendre polynomials P_k as

$$-\frac{Gm}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+\varepsilon^2}} = -Gm \sum_{k=0}^{\infty} \frac{r_<^k}{p^{k+1}} P_k(\cos\gamma'), \quad (2)$$

where

$$\cos\gamma' = \frac{r_>}{p}\cos\gamma, \quad (3)$$

and γ is the angle between $\mathbf{r}$ and $\mathbf{r}'$.

The problem becomes apparent: since the Legendre polynomials are orthogonal *only in the interval* $[-1, 1]$, and, as is shown by Eq. (3), the variable $\cos\gamma'$ does not cover all of that interval, then the set $\{P_k(\cos\gamma')\}$ is not orthogonal, and therefore the terms of Eq. (1) do not represent complete multipoles. Instead, each multipole is spread through infinite terms of the expansion.

It is easy to see this in a simple example, namely the potential at a point outside a homogeneous sphere of density ρ_0, radius R and total mass M, centered at the origin of the expansion. Since the mass distribution is monopolar, the generated potential $\Phi(\mathbf{r})$ should be completely described by the monopolar term of the expansion. But we have

$$\begin{aligned}\Phi(\mathbf{r}) &= -G\int_{|\mathbf{r}'|<R} \frac{\rho_0}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+\varepsilon^2}} \mathrm{d}^3\mathbf{r}' \\ &= -\frac{GM}{p}\left[1 - \frac{3R^2\varepsilon^2}{10p^4} - \frac{3R^4(4r^2\varepsilon^2 - 3\varepsilon^4)}{56p^8} + \cdots\right].\end{aligned} \quad (4)$$

It is clear that $-GM/p$ does not contain by itself the monopole of the potential.

2. MULTIPOLAR EXPANSION WITH SOFTENING

In order to recover complete multipoles in the expansion of the Newtonian potential with softening, we proceed as follows. Let us define for simplicity

$$s = \frac{r_>}{p}, \quad q = \frac{r_<}{p}, \quad (5)$$

and $p = \sqrt{r_>^2 + \varepsilon^2}$ as before. Thus, Eq. (2) becomes

$$-\frac{Gm}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+\varepsilon^2}} = -\frac{Gm}{p}\sum_{k=0}^{\infty} P_k(s\cos\gamma)q^k. \quad (6)$$

We are interested in the projection of the infinite sum onto each of the Legendre polynomials $\{P_k(\cos\gamma)\}$, i.e., we search for the coefficients C_n such that

$$\Phi(\mathbf{r}) = -\frac{Gm}{p}\sum_{n=0}^{\infty} C_n P_n(\cos\gamma). \quad (7)$$

We have for the monopolar coefficient:

$$\begin{aligned}C_0 &= \frac{1}{2}\int_{-1}^{1}\left[\sum_{k=0}^{\infty} P_k(s\cos\gamma)q^k\right] P_0(\cos\gamma)\mathrm{d}(\cos\gamma) \\ &= 1 + \frac{1}{s}\sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{q^k}{2k+1}(P_{k+1}(s) - P_{k-1}(s)),\end{aligned} \quad (8)$$

where we have taken advantage of the fact that the series (6) is uniformly convergent. This also allows to rearrange the last expression into a series in powers of s:

$$\begin{aligned}C_0 \;=\; 1 + \frac{1}{s}\Bigg\{ &\sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} D_{k,1}\frac{q^k}{2k+1}s \\ &+ \sum_{\substack{i=3 \\ i \text{ odd}}}^{\infty}\left[\sum_{\substack{k=i-1 \\ k \text{ even}}}^{\infty} D_{k,i}\frac{q^k}{2k+1}\right]s^i\Bigg\},\end{aligned} \quad (9)$$

where $D_{k,i}$ is the (as yet unknown) coefficient of s^i in the expression $P_{k+1}(s) - P_{k-1}(s)$. Starting from the definition (e.g., Gradshteyn & Ryzhik (1980))

$$P_k(s) = \frac{1}{2^k}\sum_{j=0}^{E(k/2)} (-1)^j \binom{k}{j}\binom{2k-2j}{k} s^{k-2j}, \quad (10)$$

where $E(z)$ stands for the integer part of z, we obtain

$$\begin{aligned}D_{k,i} = &\frac{(-1)^{(k+1-i)/2}}{2^{k-1}}\frac{2k+1}{k+1-i} \\ &\times \binom{k-1}{\frac{k-1-i}{2}}\binom{k-1+i}{k-1}.\end{aligned} \quad (11)$$

This allows us to compute

$$D_1 \equiv \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} D_{k,1}\frac{q^k}{2k+1} = \frac{1}{\sqrt{1+q^2}} - 1, \quad (12)$$

and

$$D_i \equiv \sum_{\substack{k=i-1 \\ k \text{ even}}}^{\infty} D_{k,i}\frac{q^k}{2k+1} = \frac{(2q)^{i-1}}{(1+q^2)^{i-1/2}}\frac{\Gamma(i-1/2)}{\sqrt{\pi}\Gamma(i+1)}. \quad (13)$$

Replacing Eqs. (12) and (13) into Eq. (9), we finally obtain

$$C_0 = -\frac{R_- - R_+}{2sq}, \tag{14}$$

where $R_\pm = \sqrt{q^2 \pm 2sq + 1}$.

Proceeding in a similar way, we obtain after some lengthy computations that

$$C_1 = -3\frac{R_- + R_+}{2sq} - \frac{R_-^3 - R_+^3}{2s^2q^2},$$
$$C_2 = -5\frac{R_- - R_+}{2sq} - 5\frac{R_-^3 + R_+^3}{2s^2q^2} - \frac{R_-^5 - R_+^5}{2s^3q^3}, \tag{15}$$

and, in general,

$$\begin{aligned} C_n &= \frac{2n+1}{\sqrt{\pi}} \sum_{k=0}^{n} (-1)^k \frac{\Gamma(-k-1/2)}{2^{k+1}} \left.\frac{\mathrm{d}^k P_n(s)}{\mathrm{d}s^k}\right|_{s=1} \\ &\times \frac{R_-^{2k+1} + (-1)^{n-k+1} R_+^{2k+1}}{2(sq)^{k+1}}. \end{aligned} \tag{16}$$

Since

$$\left.\frac{\mathrm{d}^k P_n(s)}{\mathrm{d}s^k}\right|_{s=1} = \frac{1}{2^k k!}\frac{(n+k)!}{(n-k)!}, \tag{17}$$

Eq. (16) may be written

$$C_n = -\sum_{k=0}^{n} b_{n,k} \frac{R_-^{2k+1} + (-1)^{n-k+1} R_+^{2k+1}}{2(sq)^{k+1}}, \tag{18}$$

where

$$b_{n,k} = \frac{2n+1}{2k+1}\binom{n+k}{2k} \tag{19}$$

is the (n,k)-th Lucas' number with odd subscript. Replacing these coefficients into Eq. (7) gives the desired multipolar expansion of the gravitational potential. However, some of these coefficients are indeterminate when $q \to 0$, so we need alternative expressions in order to work numerically. Thus, we rewrite them, obtaining after some algebra

$$C_n = 2\frac{(R_+ - R_-)^n}{(R_+ + R_-)^{n+1}}, \tag{20}$$

and the multipolar expansion of the potential is thus

$$\begin{aligned} \Phi(\mathbf{r}) &= \sum_{n=0}^{\infty} \Phi_n(\mathbf{r}) = -\frac{2Gm}{p} \\ &\times \sum_{n=0}^{\infty} \frac{(R_+ - R_-)^n}{(R_+ + R_-)^{n+1}} P_n(\cos\gamma), \end{aligned} \tag{21}$$

where Φ_n represents the n-polar term of the expansion. The accelerations can be readily computed as

$$\begin{aligned} \mathbf{a}_n &= -\nabla\Phi_n \\ &= \frac{2q}{p^2}\frac{(R_+ - R_-)^{n-1}}{(R_+ + R_-)^{n+2}} \\ &\times \left\{ \left[\left[\frac{(2n+1)(1-2s^2+q^2)}{R_+R_-} - 1 \right] P_n(\cos\gamma) \right.\right. \\ &\left. -2n\cos\gamma \frac{P_{n-1}(\cos\gamma) - \cos\gamma P_n(\cos\gamma)}{1-\cos^2\gamma} \right] \mathbf{e}_r \\ &\left. +2n\frac{P_{n-1}(\cos\gamma) - \cos\gamma P_n(\cos\gamma)}{1-\cos^2\gamma} \mathbf{e}_{r'} \right\}, \end{aligned} \tag{22}$$

where $\mathbf{e}_r = \mathbf{r}/r$ and $\mathbf{e}_{r'} = \mathbf{r}'/r'$. As a reference, we give the first four terms explicitly:

$$\begin{aligned} \mathbf{a}_0 &= -\frac{Gm}{p^3}\frac{\mathbf{r}}{s^2(R_+ + R_-)}\left(1 - \frac{1+q^2-2s^2}{R_+R_-}\right), \\ \mathbf{a}_1 &= -\frac{Gm}{p^3}\frac{4}{(R_+ + R_-)^3}\left[\left(1 - \frac{1+q^2-2s^2}{R_+R_-}\right)\right. \\ &\times \left.\frac{3q\cos\gamma}{s}\mathbf{r} + 2\mathbf{r}'\right], \\ \mathbf{a}_2 &= \frac{Gm}{p^3}\frac{8q}{(R_+ + R_-)^5}\left\{12s\cos\gamma\mathbf{r}' + q\mathbf{r}\left[1 - \right.\right. \\ &\left.\left. 15\cos^2\gamma + 5(3\cos^2\gamma - 1)\frac{1+q^2-2s^2}{R_+R_-}\right]\right\}, \\ \mathbf{a}_3 &= \frac{2Gm}{p^3}\frac{(R_+ - R_-)^2}{(R_+ + R_-)^5}\left\{6(5\cos^2\gamma - 1)\mathbf{r}'\right. \\ &+ \frac{q\cos\gamma}{s}\left[7(5\cos^2\gamma - 3)\right. \\ &\times \left.\left.\left(\frac{1+q^2-2s^2}{R_+R_-} - 1\right) - 12\right]\mathbf{r}\right\}. \end{aligned} \tag{23}$$

3. NUMERICAL RESULTS

In order to compare the gravitational potential expanded in traditional multipoles against the gravitational potential expanded as in Eq. (21), we first set the origin of the multipolar expansion at the point $(0,0,0)$, and then put a point mass $m = 1$ at $\mathbf{r}' = (1,0,0)$. We then computed, along the x axis: (a) the total softened potential $\Phi_{\mathrm{S}}(\mathbf{r}) = -Gm[(\mathbf{r}-\mathbf{r}')^2 + \varepsilon^2]^{-1/2}$; (b) the traditional multipolar expansion of the potential (Eq. (1)) up to quadrupole terms, $\Phi_{\mathrm{T2}}(\mathbf{r})$; and (c) the multipolar expansion of the potential according to Eq. (21) up to quadrupole terms, $\Phi_{\mathrm{M2}}(\mathbf{r})$. In all cases $G = 1$ was used. Figure 1 shows the results when $\varepsilon = 1$.

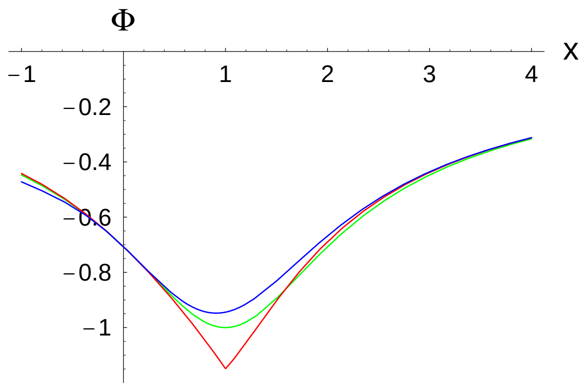

Fig. 1. Potentials $\Phi_{\rm S}$ (middle curve at $x = 1$), $\Phi_{\rm T2}$ (lower curve) and $\Phi_{\rm M2}$ (upper curve) of a particle at $\mathbf{r}' = (1, 0, 0)$, computed along the x axis. Here $\varepsilon = 1$.

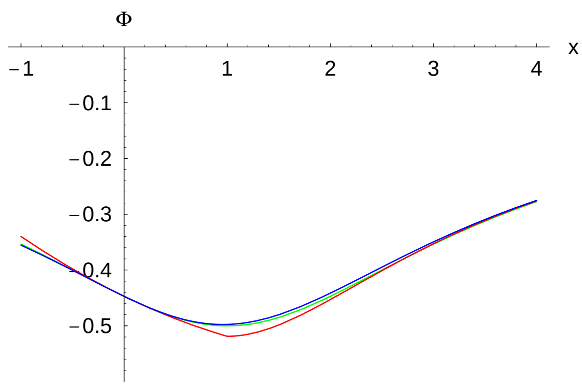

Fig. 2. Same as Fig. 1, but with a softening $\varepsilon = 2$. The lower curve at $x = 1$ corresponds to $\Phi_{\rm T2}$.

Clearly, near m, the complete monopole, dipole and quadrupole provided by $\Phi_{\rm M2}$ result in a slightly better representation of the total potential. Farther away, the three potentials are essentially the same.

Figure 2 shows the result of doubling the softening. The softening being so large, the resulting (total) potential is almost monopolar: near the origin of coordinates, ε dominates, whereas far from the origin the particle is seen almost centered (as with any other ε). As a result, $\Phi_{\rm M2}$ reproduces very well the total potential, and $\Phi_{\rm T2}$, having only part of the monopole, does a little less good a job. Halving ε from its primitive value, results in the potentials shown in Figure 3. In this case, the presence of higher multipoles is conspicuous; $\Phi_{\rm M2}$, containing only up to quadrupole terms, cannot reproduce the potential near m. Notably, $\Phi_{\rm T2}$ can do it. This demonstrates again that $\Phi_{\rm T2}$ is not a sum of up to

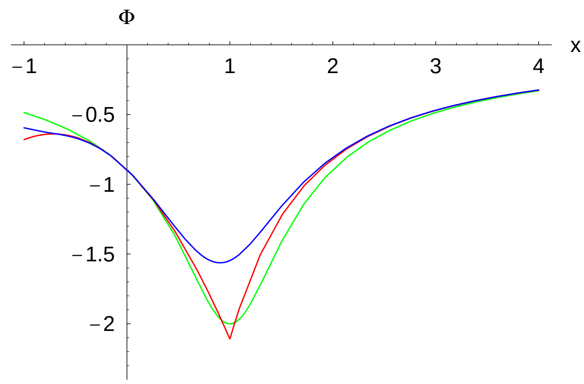

Fig. 3. Same as Fig. 1, but with a softening $\varepsilon = 0.5$. The lower curve at $x = 1$ corresponds to $\Phi_{\rm T2}$, the middle curve to $\Phi_{\rm S}$, and the upper curve to $\Phi_{\rm M2}$.

quadrupole terms (although in this case this lack of completeness favours the result!).

These results lead to the question of whether an N-body code using a multipolar expansion is actually affected by these differences. We therefore examined the effect of using $\Phi_{{\rm M},l}$ (i.e., Eq. (21) up to l-pole terms) instead of $\Phi_{{\rm T},l}$ (Eq. (1) up to l-pole terms) in a tree code, where a particle is accelerated by its neighbours through the potential $\Phi_{\rm S}$, and far particles are clustered in groups, each group contributing to the acceleration through $\Phi_{{\rm T},l}$. The clustering is done whenever the ratio of the size of the group to the distance to the group is less than a certain threshold θ, called the opening angle. Following Barnes & Hut (1989), Hernquist & Barnes (1990), and Dyer & Ip (1993), we set up an equilibrium, isotropic King (1966) model with central potential $\Phi(\mathbf{0}) = -5\sigma^2$, made up of $N = 100,000$ particles. The units were chosen such that $G = 1$, the total mass $M = 1$, and the rms velocity $\langle v^2 \rangle = 1$; if $\varepsilon = 0$, this leads to a total energy of the system $E = -1/2$. For each particle, three accelerations were computed: (a) the acceleration $\mathbf{a}_{\rm PP}$ resulting from a particle-particle interaction (with potential $\Phi_{\rm S}$) with the rest of the particles. This is the acceleration against which the other two will be compared; (b) the acceleration $\mathbf{a}_{\rm TC}$ resulting from a tree code-like interaction (with potentials $\Phi_{\rm S}$ and $\Phi_{{\rm T},l}$) with the rest of the particles, using an opening angle $\theta = 0.7$; (c) the acceleration $\mathbf{a}_{\rm M}$ computed as in (b), but using the expressions (23) derived from $\Phi_{{\rm M},l}$ whenever the accelerations of $\Phi_{{\rm T},l}$ would have been used in (b). Figure 4 shows the relative error in using $\mathbf{a}_{\rm TC}$ instead of $\mathbf{a}_{\rm PP}$ versus the relative error in using $\mathbf{a}_{\rm M}$ instead of $\mathbf{a}_{\rm PP}$, varying the softening ($\varepsilon = 0.0125$

in the first column, $\varepsilon = 0.025$ in the second one, and $\varepsilon = 0.05$ in the third one) and the number l of terms included in the multipolar expansion (only the monopole $l = 0$ in the first row, $l = 0$ and $l = 2$ in the second row, and $l = 0$, $l = 2$, and $l = 3$ in the third row). As can be seen, the error is effectively reduced in most particles, and, except for the smallest softening, the correction is significant in all the cases. This may be explained as follows: although a set of particles scattered without any apparent symmetry (as is the case for most grouped particles in a tree code) should generate a potential composed of many multipoles, their potential is being computed at a point far away from them, i.e., where the high multipoles are already damped. As only the low order multipoles are strongly contributing to the potential, $\mathbf{a}_{\rm M}$ has the chance of giving a better approximation to $\mathbf{a}_{\rm PP}$ than $\mathbf{a}_{\rm TC}$. On the other hand, Fig. 4 shows that the differences between $\mathbf{a}_{\rm TC}$ and $\mathbf{a}_{\rm M}$ become more noticeable as ε grows, and also when more terms are added to the multipolar expansion. These differences are quite impressive already with a moderate value of the softening and a quadrupolar expansion.

One may therefore wonder whether this has any noticeable influence on the evolution of the system. To answer this, we performed a series of experiments in which King spheres with $\Phi(\mathbf{0}) = -5\sigma^2$ and N = 5000, 10,000, 20,000, and 50,000 were evolved during 20 dynamical times using a standard tree code. In each case, several fixed time steps were used: Δt = 0.005, 0.02, and 0.05. For each experiment, three runs were performed: one using $\theta = 0$, which is equivalent to a particle-particle integration; one using $\theta = 0.7$ with a traditional expansion $\Phi_{\rm T,2}$ (i.e., up to quadrupole terms), and a third one with the same opening angle in which the multipolar expansion $\Phi_{\rm M,2}$ was used instead of $\Phi_{\rm T,2}$. In order to keep a moderate softening but at the same time to obtain non-negligible corrections, a value of $\varepsilon = 0.025$ was used throughout. Except when $\Delta t = 0.05$, the relative energy conservation was better than 6×10^{-4} in all the experiments; with the larger time step, the relative energy conservation was better than 8×10^{-3}. (There were no significant differences in energy conservation when varying the method of integration, i.e., particle-particle or either of the multipolar expansions.) In order to detect any differences between the three methods, a gauge which is sensitive to the details of the accelerations should be used; the density profile, being a differential feature, was chosen to that end. Thus, for each run, a density profile was generated at the end of the integration. It is known that the computation of a reliable density profile is tricky (Ascasibar & Binney 2005), and specially in spherical coordinates (Merritt & Tremblay 1994); in fact, *any* density profile obtained from discrete data is a major task (see, e.g., Silverman 1998). We computed all density profiles using several methods, from the most simple ones as: (a) histograms, (b) counts of particles in fixed spherical shells, and (c) computation of volumes occupied by a fixed number of particles (nearest n-th neighbour method), to the more sophisticated ones, as (d) first computing a fixed three-dimensional kernel from a Gaussian one-dimensional one, thus avoiding the problem of the origin of coordinates (based on Merritt & Tremblay (1994), who solved this problem for a two-dimensional case), and then using a fixed window chosen by eye (Merritt & Tremblay 1994) to compute the density; (e) the same, but computing an optimal window by maximizing the likelihood cross-validation function (Silverman 1998), (f) the same, but using an adaptive kernel based on local bandwidth factors (Silverman 1998), and (g) the same, but using an adaptive kernel based on the nearest n-th neighbours. There were no essential differences between the results obtained with either method; we chose to display the density computed from spherical shells simply because it is a good compromise between noise and bias.

Figure 5 shows the case for $N = 50,000$ and $\Delta t = 0.02$. As can be seen, the three profiles overlap almost entirely; there were no essential differences in using one expansion or the other. The same turned out to be the case for the rest of the experiments, although with noisier profiles for the smaller Ns as expected. These results are probably a consequence of $\Phi_{\rm T2}$ and $\Phi_{\rm M2}$ being noticeably different only inside a sphere of radius $\simeq \varepsilon$ around each particle, whereas the multipolar expansion in a tree code is applied only when particles are far apart.

Figure 6 helps to clear up this point. The inset shows a histogram of the differences between $\mathbf{a}_{\rm TC}$ and $\mathbf{a}_{\rm M}$ corresponding to the central panel of Fig. 4 (note that in this case there is no normalization). The main panel shows this same histogram, but with a scale in which the root mean squared (particle-particle, or total) acceleration $\langle a^2\rangle^{1/2} = 2.87$ can be plotted (a double arrow points to this value in the figure); the histogram is barely seen at the left. This clearly shows that the absolute values of the differences between accelerations are in general negligible with respect to the values of the accelerations themselves, supporting the hypothesis given above.

Still, the particles in the examples examined, being in an equilibrium state, do not suffer large accel-

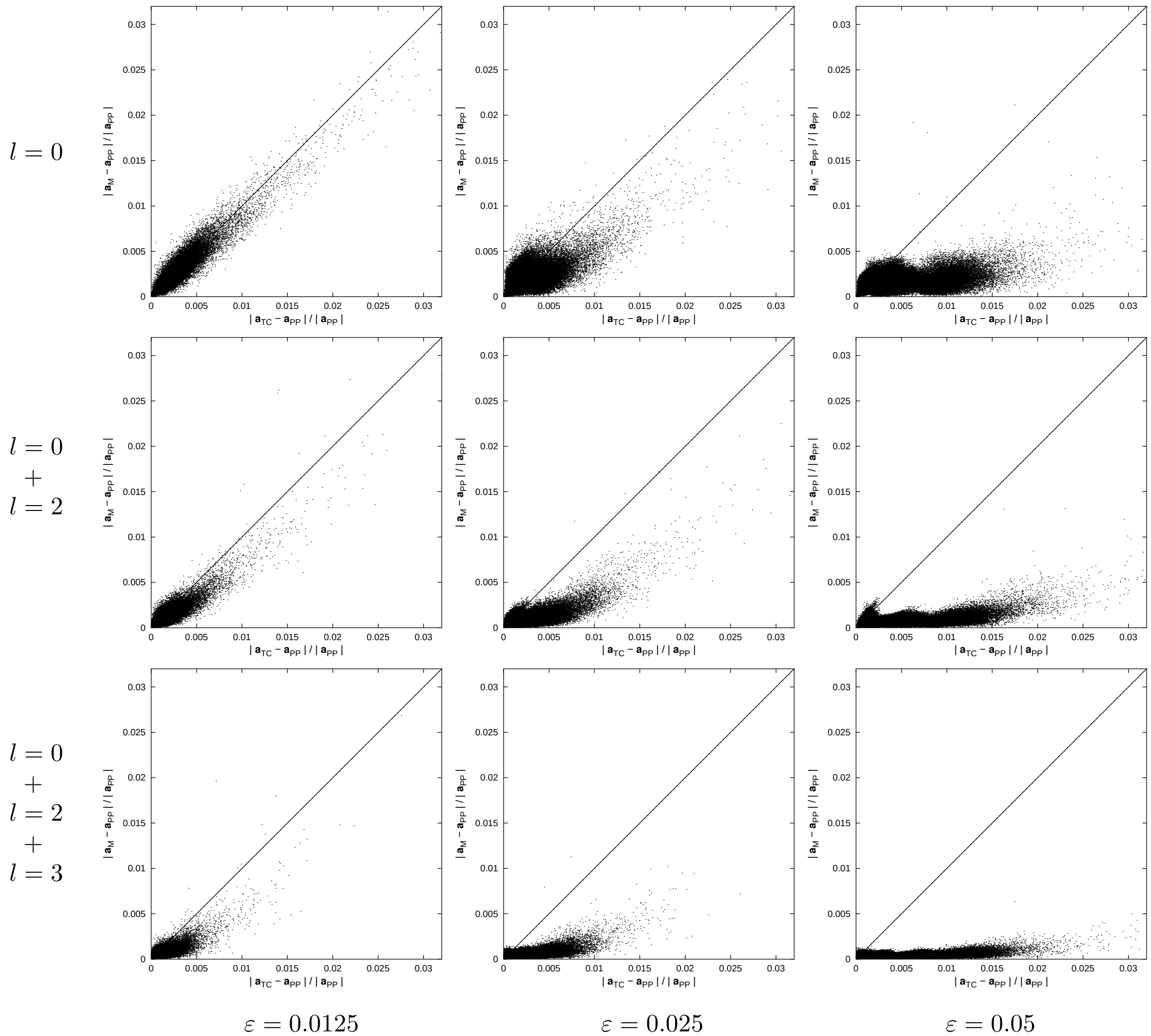

Fig. 4. Relative error in acceleration (with respect to a particle-particle (PP) computation) using the complete multipolar expansion (M), versus the same but using the traditional multipolar expansion (TC), at different values of the softening and with several multipolar truncations.

erations in general. But when large accelerations are present, small percentage corrections to them might have a non-negligible effect. To probe a regime of large accelerations, we set up a 10,000-body sphere with initial density $\propto r^{-1}$, radius $R = 1$, and null velocities, and let it collapse, computing the accelerations of each particle by the three above mentioned methods. We used a time step of 0.02 initial crossing times, and a softening $\varepsilon = 0.025$. The integrations were followed during 50 crossing times, i.e., well after the systems have reached a quasi-equilibrium state. The relative total energy was conserved to better than 8×10^{-4} in the three cases. Table 1 shows the final values of the kinetic energy T, the potential energy W, the virial ratio $2T/|W|$, the total energy E and the number of escapees N_d for the three simulations. As can be seen, the global parameters of the collapses are essentially the same, although the values obtained using $\mathbf{a}_{\rm M}$ were slightly closer to the particle-particle integration than those obtained using $\mathbf{a}_{\rm TC}$. As before, we computed the final density profiles in order to compare differential features of the systems. In this case, care was taken of the shifting of the center of density with respect to the center of mass caused by escapees, by using a friend-of-friend algorithm (i.e., an algorithm to recognize gravitationally bounded groups in an N-body system).

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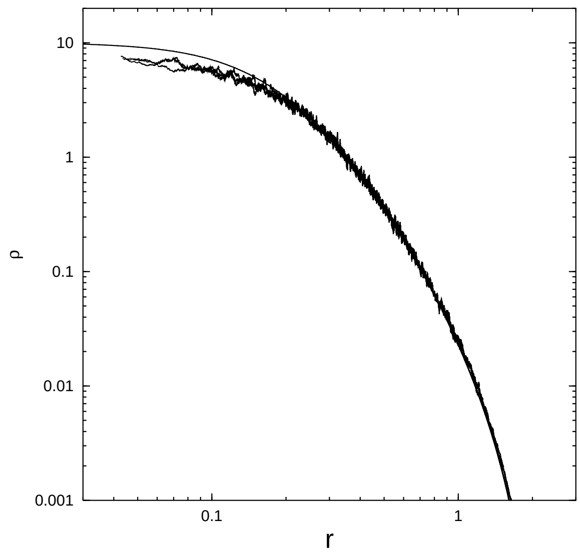

Fig. 5. Final density profiles of a King sphere, evolved using the particle-particle accelerations $\mathbf{a}_{\rm PP}$, with tree code-like interactions using $\mathbf{a}_{\rm TC}$, and with tree code-like interactions using $\mathbf{a}_{\rm M}$. The theoretical profile is shown as a reference.

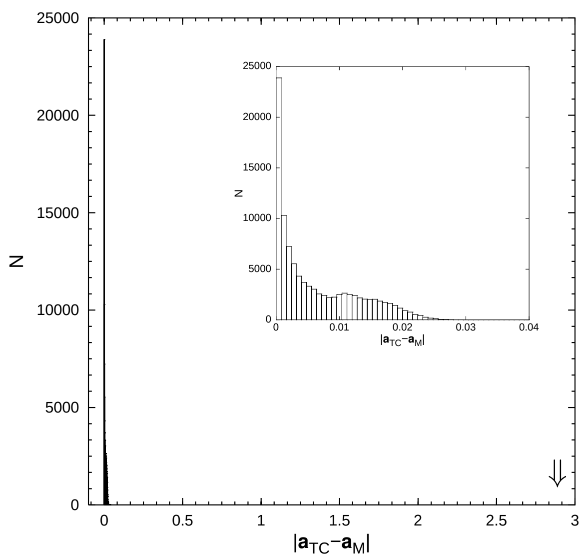

Fig. 6. Histogram of the differences between $\mathbf{a}_{\rm TC}$ and $\mathbf{a}_{\rm M}$ for the King sphere of the central panel of Fig. 4, with a scale in which they are visible (inset) and with a scale in which the root mean squared acceleration is visible (see text).

Figure 7 shows the outcome: the three profiles are virtually indistinguishable, confirming that there is no dynamical influence in using one or another

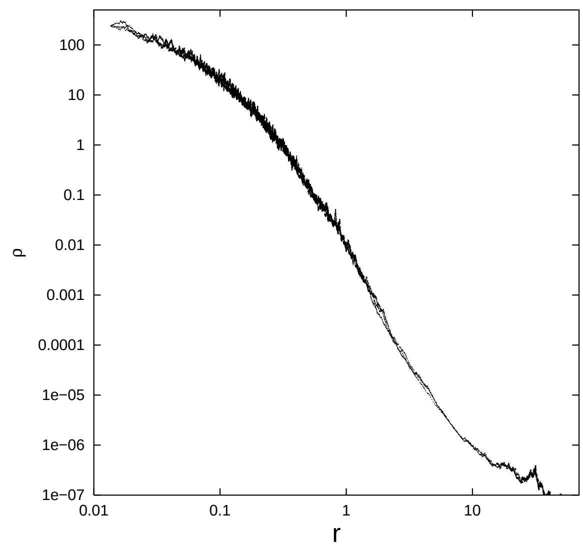

Fig. 7. Final density profiles of a cold collapse, evolved using the particle-particle accelerations $\mathbf{a}_{\rm PP}$, with tree code-like interactions using $\mathbf{a}_{\rm TC}$, and with tree code-like interactions using $\mathbf{a}_{\rm M}$. The three profiles are virtually the same.

multipolar expansion in a tree code. We also performed an additional experiment with a time step of 0.01 initial crossing times, and another with 20,000 particles (although in this latter case we followed the evolution only up to 20 crossing times, the final system being anyway in a quasi steady-state). These additional experiments did not show differences either.

It is to be noted that, besides these results showing that a change of expansion is not worth the effort, there is another reason not to change the traditional expansion: neither Eq. (21) nor Eqs. (23) can be neatly divided in a part containing only $\mathbf{r}$ and another part containing only $\mathbf{r}'$; thus, the computation of the acceleration in a tree code-like way is of order N^2, the same as in a particle-particle approach, losing the property of being an $O(N \log N)$ code.

4. MULTIPOLAR NOMENCLATURE

In standard multipolar nomenclature, successive terms of the expansion of the potential are called the monopole ($l = 0$), the dipole ($l = 1$), the quadrupole ($l = 2$), the octupole ($l = 3$), and so on. It is also possible to find the names of even higher order terms in the literature (see, e. g., Barnes & Hut 1989), namely the $l = 4$ and $l = 6$ terms, called hexadecapole (16-pole) and hexacontatetrapole (64-pole), respectively. The name given to each l-pole suggests

TABLE 1

FINAL PARAMETERS OF THE SIMULATIONS OF COLD COLLAPSES

Model	T	W	$2T/\|W\|$	E	N_d
$\mathbf{a}_{\rm PP}$	0.73316	−1.3970	1.0496	−0.66385	1920
$\mathbf{a}_{\rm TC}$	0.74219	−1.4054	1.0562	−0.66323	1911
$\mathbf{a}_{\rm M}$	0.72610	−1.3900	1.0447	−0.66393	1921

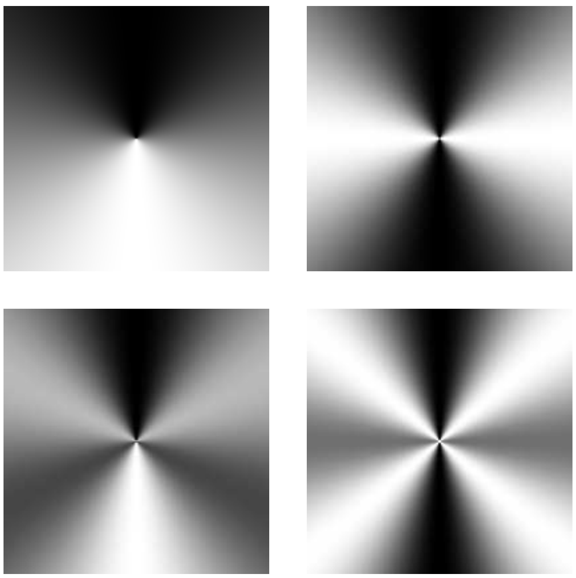

Fig. 8. Density plot of $P_l(\cos\gamma)$, where γ is the angle measured from the upper vertical direction towards both left and right. Top left: $l = 1$; top right: $l = 2$; bottom left: $l = 3$; bottom right: $l = 4$. Darker zones correspond to higher values; brighter zones to lower values.

the assumption of a 2^l-fold symmetry along the angle of the argument of the Legendre polynomials, as is the case for the first three multipoles. However, the $l = 3$ multipole has three maxima and three minima along this angle, i.e., a six-fold symmetry, as can be verified in Figure 8; in fact, any l-pole has a $2l$-fold symmetry. Therefore we consider that the traditional nomenclature induces the wrong picture and we suggest to change it in order to be consistent with the true geometry involved. So, the $l = 3$ multipole should be called the *hexapole*; the $l = 4$, octupole; the $l = 5$, decapole; the $l = 6$, dodecapole, an so on. Thus, we claim that, with the exception of the monopole, the nomenclature should follow prefixes which denote factors of 2 and not powers of 2.

5. CONCLUSIONS

We have shown how the traditional multipolar expansion of a softened gravitational potential does not isolate the multipoles as in the Newtonian case. Instead, the multipoles become spread through an infinity of terms, and it is wrong to claim the inclusion of a certain l-polar contribution when the terms of Eq. (1) are used. These claims are common in the literature, probably because of the unawareness of this feature. An expansion in complete multipoles of a softened potential is presented.

Fortunately, the overall effect of using incomplete multipoles instead of complete ones turns out to be negligible whenever the potential expansion is computed only far from its source, as is the case of tree codes used to follow the evolution of an N-body system. In this case, moreover, the computation of complete multipoles of a system of N particles turns out to be of $O(N^2)$, which makes unworthy its implementation anyway. However, since the spreading of multipoles through the terms of the expansion was not mentioned in the past, we found it important to establish to what extent it is necessary to take it into account. Also, the expansion presented here could even be used, for instance, as a gauge to identify the degree of error being made by the standard approach.

Finally, we claim that the nomenclature in multipolar expansions is misleading: the so-called 'octupole', which supposes an eight-fold symmetry because of its name, is actually a six-fold distribution (three maxima and three minima), and therefore should be called 'hexapole'. In general, an l-pole has a $2l$-fold symmetry, and this number should be the one which generates the prefix used for the name of multipoles.

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