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GRAVITOMAGNETISM AND ANGULAR MOMENTA OF BLACK-HOLES

Marcelo Samuel Berman

Instituto Albert Einstein, Curitiba, PR, Brazil

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RESUMEN

Revisamos las fórmulas del contenido energético de los hoyos negros de Kerr-Newman, para los cuales la parte de energía gravitomagnética entra en escena (Berman 2004, 2006a,b). Después, obtenemos las fórmulas de los momentos angulares, incluyendo el efecto gravitomagnético. Se establecen tres teoremas: (1) Ningún hoyo negro tiene su energía confinada en su interior; (2) Los hoyos negros en rotación no poseen su momento angular confinado; (3) La densidad de energía de un hoyo negro no está confinada interiormente. La diferencia entre nuestros resultados y los previamente publicados por Virbhadra (1990a,b,c), y Aguirregabiria et al. (1996), se desprende de la ausencia de energía auto-gravitacional en el cálculo efectuado por aquellos autores. Esta ausencia se nota directamente en el cálculo de la energía de un hoyo negro estático.

ABSTRACT

We review the energy contents formulae of Kerr-Newman black-holes, where gravitomagnetic energy term comes into play (Berman 2004, 2006a,b). Then, we obtain the angular momenta formulae, which include the gravitomagnetic effect. Three theorems can be enunciated: (1) No black-hole has its energy confined to its interior; (2) Rotating black-holes do not have confined angular momenta; (3) The energy density of a black-hole is not confined to its interior. The difference between our calculation and previous ones by Virbhadra (1990a,b,c), and Aguirregabiria et al. (1996), lies in the fact that we include a term responsible for the self-gravitational energy, while the cited authors discarded such effect, which appears in the static black hole energy calculation.

Key Words: **BLACK HOLE PHYSICS — GRAVITATION — MAGNETIC FIELDS**

1. INTRODUCTION

The calculation of energy and angular momentum of black-holes has, among others, an important astrophysical rôle, because such objects remain the ultimate source of energy in the Universe, and the amount of angular momentum is related to the possible amount of extraction of energy from the b.h. (Levinson 2006).

In a series of excellent papers, Virbhadra (1990a,b,c) and Aguirregabiria, Chamorro, & Virbhadra (1996) calculated the energy contents, as well as the angular momenta, for Kerr-Newman black-holes. Notwithstanding the high quality of those papers, Berman (2004, 2006a,b) has pointed out that their results for the energy do not reduce to the correct well known result by Adler, Bazin, & Schiffer (1975), when the electric charge and rotation parameters go to zero. Furthermore, Berman (2004, 2006a,b) objected the energy formula (6) and (8) below obtained by Virbhadra (1990a,b,c) and Aguirregabiria et al. (1996), because the gravitomagnetic effect on the energy contents of the Kerr-Newman black-hole does not appear in their results. Soon afterwards, Ciufolini & Pavlis (2004) and Ciufolini (2004) reported the experimental verification of the Lens-Thirring effect. This effect is a consequence of the concept of gravitomagnetism.

Therefore, it is now interesting to check whether the calculation of angular momenta contents for a K.N. black hole given by Virbhadra (1990a,b,c) and Aguirregabiria et al. (1996) includes the gravitomagnetic contribution. It will be seen that this does not occur. We recalculate here the angular momenta formulae, in order that gravitomagnetism enters into the scenario. We cite in our favor the papers by Lynden-Bell & Katz (1985) and Katz & Ori (1990).

2. CALCULATION OF ENERGY AND ANGULAR MOMENTA

The metric for a K.N. black hole may be given in Cartesian coordinates by:

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2\left[M - \frac{Q^2}{2r_0}\right] r_0^3}{r_0^4 + a^2 z^2} \cdot F^2 \quad , \tag{1}$$

while

$$F = dt + \frac{Z}{r_0} dz + \frac{r_0}{(r_0^2 + a^2)} (xdx + ydy) + \frac{a\,(xdy - ydx)}{a^2 + r_0^2} \quad , \tag{2}$$

$$r_0^4 - \left(r^2 - a^2\right) r_0^2 - a^2 z^2 = 0 \quad , \tag{3}$$

and

$$r^2 \equiv x^2 + y^2 + z^2 \quad . \tag{4}$$

In the above, M, Q, and "a" stand respectively for the mass, electric charge, and the rotational parameter, which has been shown to be given by:

$$a = \frac{J_{TOT}}{M} \quad , \tag{5}$$

where J_{TOT} stands for the total angular momentum of the system.

According to Virbhadra (1990a,b,c) and Aguirregabiria et al. (1996), the energy-momentum pseudo tensor is given by:

$$\bar{P}_0 = M - \left[\frac{Q^2}{4\varrho}\right] \left[1 + \frac{\left(a^2 + \varrho^2\right)}{a\varrho} arctgh\left(\frac{a}{\varrho}\right)\right] \quad , \tag{6}$$

$$P_1 = P_2 = P_3 = 0 \quad . \tag{7}$$

It must be remarked that when we calculate, in this paper, energy or angular momentum contents, we suppose that the calculation is done over a closed surface with constant ρ; the total energy and the total angular momentum should be calculated in the limit $\rho \to \infty$.

If we expand relation (6) in powers of $(\frac{a}{\rho})$, and retain only the first and second powers, we obtain the approximate relation, valid for slow rotational motion:

$$\bar{E} = \bar{P}_0 \cong M - \left[\frac{Q^2}{R}\right] \left[\frac{a^2}{3R^2} + \frac{1}{2}\right] \quad , \tag{8}$$

where $\varrho \to R$; this can be seen because the defining equation for ϱ is:

$$\frac{x^2 + y^2}{\varrho^2 + a^2} + \frac{z^2}{\varrho^2} = 1 \quad \text{and if} \quad a \to 0, \quad \varrho \to R\,. \tag{9}$$

In the same token, the cited authors obtained, for angular momentum, defined by:

$$J^{(3)} = \int \left[x^1 p_2 - x^2 p_1\right] d^3x \quad , \tag{10}$$

and for the above metric,

$$J^{(1)} = J^{(2)} = 0 \quad , \tag{11}$$

where p_i stand for the linear momentum densities ($i = 1, 2, 3$).

The cited authors also found:

$$\bar{J}^{(3)} = aM - \left[\frac{Q^2}{4\varrho}\right] a \left[1 - \frac{\rho^2}{a^2} + \frac{\left(a^2+\varrho^2\right)^2}{a^3\varrho} arctgh\left(\frac{a}{\varrho}\right)\right] \quad , \tag{12}$$

and when we go to the slow rotation case, Virbhadra found:

$$\bar{J}^{(3)} \cong aM - 2Q^2 a \left[\frac{a^2}{5R^3} + \frac{1}{3R}\right] \quad . \tag{13}$$

Unfortunately, when $Q = 0$ in the above equations, we are left without gravitomagnetic effects. The corrections made by Berman (2004, 2006a,b), in the above cited papers, reside on the acknowledgment that due to the same R^{-2} dependence of the gravitation and electric interactions, as characterized by Newton's law of gravitation, and Coulomb's law for electric charges, we would have on a par the equal contributions of charge and mass in the above formulae, so that, except for the inclusion of the inertial term, we should make the correction:

$$Q^2 \to Q^2 + M^2 \quad . \tag{14}$$

In corroboration of correction (14), we cite that, for the Reissner-Nordström metric, the energy formula is given by:

$$E_{RN} = M - \left[\frac{Q^2+M^2}{2R}\right] \quad . \tag{15}$$

Relation (15) reduces correctly to the energy formula published by Adler et al. (1975) for the spherical mass distribution, when we make $Q = 0$ in (15). On the contrary, relations (6) and (8) do not reduce to Adler et al.'s formula when $a = Q = 0$, nor to the relation (15), when $a = 0$ in relation (8). This shows that correction (14) is plausible. In fact, by applying pseudo tensors,

$$P_\mu = \int_t \sqrt{-g}\left[T^0_\mu + t^0_\mu\right] d^3x \;=\; \text{constants} \quad , \tag{16}$$

where

$$\sqrt{-g}t^V_\mu = \frac{1}{2\mathbf{C}}\left[U g^\nu_\mu - \frac{\partial U}{\partial g^{\pi\beta}_{|\nu}} g^{\pi\beta}_{|\mu}\right] \quad , \tag{17}$$

$$U = \sqrt{-g}g^{\rho\sigma}\left[\left\{\begin{array}{c}\alpha\\ \sigma\varrho\end{array}\right\}\left\{\begin{array}{c}\beta\\ \alpha\beta\end{array}\right\} - \left\{\begin{array}{c}\alpha\\ \beta\varrho\end{array}\right\}\left\{\begin{array}{c}\beta\\ \alpha\sigma\end{array}\right\}\right] \quad , \tag{18}$$

and

$$\mathbf{C} = -\frac{8\pi G}{c^2} \quad , \tag{19}$$

we find the correct relations for the energy and momentum:

$$P_0 = M - \left[\frac{Q^2+M^2}{4\varrho}\right]\left[1 + \frac{\left(a^2+\varrho^2\right)}{a\varrho} arctgh\left(\frac{a}{\varrho}\right)\right] \quad , \tag{20}$$

$$P_1 = P_2 = P_3 = 0 \quad . \tag{21}$$

The last result "validates" the coordinate system chosen for the present calculation: it is tantamount to the choice of a center-of-mass coordinate system in Newtonian Physics, or the use of comoving observers in Cosmology.

By considering an expansion of the arctgh $(\frac{a}{\varrho})$ function, in terms of increasing powers of the parameter "a", and by neglecting terms $a^3 \simeq a^4 \simeq .... \simeq 0$, we find the energy of a slowly rotating Kerr-Newman black-hole,

$$E = \bar{P}_0 \simeq M - \left[\frac{Q^2 + M^2}{R}\right]\left[\frac{a^2}{3R^2} + \frac{1}{2}\right] \quad , \tag{22}$$

where $\varrho \rightarrow R$.

We can interpret the terms $\frac{Q^2 a^2}{3R^3}$ and $\frac{M^2 a^2}{3R^3}$ as the magnetic and gravitomagnetic energies caused by rotation. Virbhadra (1990a,b,c) and Aguirregabiria et al. (1996) noticed the first of these effects in the year 1990, but since then it seems that they failed to recognize the existence of the gravitomagnetic energy due to M, on an equal footing.

Likewise, if we apply Equation 10:

$$J^{(3)} = \int \left[x^1 p_2 - x^2 p_1\right] d^3x \quad ,$$

where the linear momentum densities are given by:

$$p_1 = -2\left[\frac{\left(Q^2 + M^2\right)\rho^4}{8\pi(\rho^4 + a^2 z^2)^3}\right] a y \rho^2 \quad , \tag{23}$$

$$p_2 = -2\left[\frac{\left(Q^2 + M^2\right)\rho^4}{8\pi(\rho^4 + a^2 z^2)^3}\right] a x \rho^2 \quad , \tag{24}$$

$$p_3 = 0 \quad , \tag{25}$$

while the energy density is given by:

$$\mu = \left[\frac{\left(Q^2 + M^2\right)\rho^4}{8\pi(\rho^4 + a^2 z^2)^3}\right]\left(\rho^4 + 2a^2\rho^2 - a^2 z^2\right) \quad , \tag{26}$$

we find:

$$J^{(3)} = aM - \left[\frac{Q^2 \; + \; M^2}{4\varrho}\right] a \left[1 - \frac{\rho^2}{a^2} + \frac{\left(a^2 + \varrho^2\right)^2}{a^3\varrho} arctgh\left(\frac{a}{\varrho}\right)\right] \quad . \tag{27}$$

Expanding the *arctgh* function in powers of $(\frac{a}{\varrho})$, and retaining up to third power, we find the slow rotation angular momentum:

$$J^{(3)} \cong aM - 2\left[Q^2 + M^2\right] a \left[\frac{a^2}{5R^3} + \frac{1}{3R}\right] \quad . \tag{28}$$

In the same approximation, relation (26) would become:

$$\mu \cong \left[\frac{Q^2 + M^2}{4\pi R^4}\right]\left[\frac{a^2}{R^2} + \frac{1}{2}\right] \quad . \tag{29}$$

The above formula could be also found by applying directly the definition,

$$\mu = \frac{dP_0}{dV} = \frac{1}{4\pi R^2}\frac{dP_0}{dR} \quad . \tag{30}$$

where P_0 would be given by the approximation (22), with $P_0 = E$ (Berman 2004, 2006a,b).

3. FINAL COMMENTS AND CONCLUSIONS

The different approach in our paper, as compared with those of Virbhadra (1990a,b,c), and Aguirregabiria et al. (1996), can be recognized from the lack of a self-gravitational energy term in those authors' calculations; for instance, Adler et al. (1975) calculated the self-gravitational-energy of a spherical mass distribution by the term $-\frac{GM^2}{2R}$. However, we cannot trace this term in their formulae (6) and (8); it is present in our calculation, as in formulae (20) and (22). Except for the inertial mass-energy term M, the self-gravitational and self-electric energies, in our calculation, present similar contributions which, for the static black hole, are given by $(-\frac{GM^2}{2R})$ and $(-\frac{Q^2}{2R})$. This means that where the cited authors worked only with an electric term $-\frac{Q^2}{2R}$, we must work with both mass and charge contributions.

We recollect now a series of statements that we have shown above to be incorrect, and which appeared in the papers by Virbhadra (1990a,b,c) and Aguirregabiria et al. (1996):
(a) no angular momentum is associated with the exterior in Kerr's metric;
(b) no energy is shared by the exterior of the Kerr black hole;
(c) the energy density in the Kerr black hole equals zero;
(d) the energy density in the Schwarzschild's black hole equals zero;
(e) the entire energy of Schwarzschild's black hole is confined to its interior.

Instead, three correct statements are issued by us:
(1) No black-hole has its energy confined to its interior;
(2) Rotating black-holes do not have confined angular momenta;
(3) The energy density of a black-hole is not confined to its interior.

We further conclude that we may identify the gravitomagnetic contribution to the energy and angular momentum of the K.N. black hole, for the slow rotating case, as:

$$\Delta E \cong -\frac{M^2a^2}{3R^3} \quad , \tag{31}$$

and

$$\Delta J \cong -2M^2\left[\frac{a^3}{5R^3}+\frac{a}{3R}\right] \approx -\frac{2M^2a}{3R} \quad , \tag{32}$$

as can be checked from relations (8) and (28) above.

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Marcelo Samuel Berman: Instituto Albert Einstein, Av. Candido Hartmann, 575, No. 17, 80730-440, Curitiba-PR, Brazil (msberman@institutoalberteinstein.org).