Revista Mexicana de Astronomía y Astrofísica, **53**, 199–217 (2017)

ANALYSIS OF HANSEN'S INFERIOR AND SUPERIOR PARTIAL ANOMALIES AND THE DIVISION OF THE ELLIPTIC ORBIT INTO TWO SEGMENTS

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Received January 6 2017; accepted March 14 2017

ABSTRACT

In this paper, a novel analysis was established to prove how Hansen's inferior and superior partial anomalies k and k_1 can divide the elliptic orbit into two segments. The analysis depends on the departures of r (for k) and $1/r$ (for k_1) from their minima. By these departures, we can find: (i) Transformations relating the eccentric anomaly to k and the true anomaly to k_1. (ii) Expressions for k and k_1 in terms of the orbital elements. (iii) The interpretation and the intervals of definition of two moduli (X, S) related to k and k_1. (iv) The extreme values of r and the elliptic equations in terms of k and k_1. (v) For r' and r'', the modulus X as a measure of the asymmetry of r' (or r'') from r'' (or r'), and the modulus S_{12} as a measure of the asymmetry of r' and r'' from the minimum value of r. (vi) A description of the segments represented by k and k_1. (vii) The relative position of the radius vector at $k = 0°$ and $k_1 = 180°$.

RESUMEN

Presentamos un nuevo análisis para demostrar que las anomalías parciales superior e inferior de Hansen, k y k_1, pueden dividir a la órbita elíptica en dos segmentos. El análisis depende de qué tanto se alejan r (para k) y $1/r$ (para k_1) de sus mínimos. Con estas diferencias podemos encontrar lo siguiente. (i) Transformaciones que relacionan a la anomalía excéntrica con k y a la anomalía verdadera con k_1. (ii) Expresiones para k y k_1 en términos de los elementos orbitales. (iii) La interpretación y los intervalos de definición para los módulos X y S relacionados con k y k_1. (iv) Los valores extremos de r y las ecuaciones elípticas en términos de k y k_1. (v) Para r' y r'' el módulo X como una medida de la asimetría de r' (o bien r'') respecto de r'' (o bien r'), mientras que el módulo S_{12} como una medida de la asimetría de r' y r'' respecto al valor minimo de r. (vi) Una descripción de los segmentos representados por k y k_1. (vii) La posición relativa del radio vector en $k = 0°$ y $k_1 = 180°$.

Key Words: celestial mechanics — comets: general — methods: analytical

1. INTRODUCTION

The conventional methods of treating astronomical perturbations do not yield manageable series solutions for the motions of highly eccentric orbits (e.g. most comets and some asteroids) because they lie partly inside and partly outside the orbits of the disturbing bodies. Consequently, in applications of the conventional methods of expansion the disturbing force becomes a highly oscillating function, and results in divergent or at least very slowly convergent series expansions.

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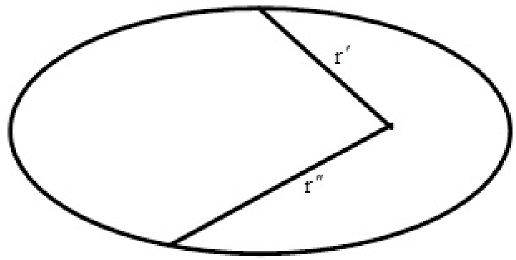

Fig. 1. Segmentation of elliptic orbit.

In an effort to overcome this situation, Hansen [1856] devised a method of computing the absolute perturbation of a periodic comet with large eccentricity based on the so called *partial anomalies*. This method involved division of the elliptic orbit of the perturbed body into segments. In each of the segments the classical variables (the true, eccentric or mean anomalies), were substituted by new ones: partial anomalies. The series representing the disturbing function was strongly convergent within the segment but invalid outside of it.

The first person to make full use of Hansens original method of partial anomalies was Nacozy (1969), who completed Hansens numerical example and compared the results with a numerical integration extending through 50 years. In his work, Nacozy utilized the pure harmonic analysis technique. In addition, the method was applied to the calculation of the general perturbations caused by Saturn on comet P/Tuttle (Skripnichenko 1972). All his analytical calculations were carried out by manipulating of Fourier series with numerical coefficients. In 1982, Sharaf proposed a regularization approach based on the idea of the orbit segmentation.

Originally, Hansen introduced two partial anomalies, the *inferior anomaly* denoted by k and the *superior anomaly* denoted by k_1. By means of these anomalies, the ellipse could be divided into two segments (as will be shown latter). Now one may ask: is it possible to divide the elliptic orbit into an arbitrary number of segments? The answer is yes, and can it be achieved firstly by a full understanding of the idea of the division of the elliptic orbit into two segments. The present paper is devoted towards this goal.

The idea of the segmentation may be stated as follows. As shown in Figure 1, let r' be the radius to a point on the orbit between periapsis and apoapsis on one side of the major axis, where $0° \leq E \leq 180°$, and let r'' be the radius to a point on the other side of the major axis, where $180° \leq E \leq 360°$.

For the segment of the orbit containing the periapsis, we may consider the *departure* of r from $r_{min} = a(1-e)$. This departure should satisfy the following conditions.
1. To be a periodic function of one independent variable.
2. To be positive $\forall r$ between r' and r''.
3. To attain its maximum values at r' and r'', and its minimum value at $a(1-e)$.
This departure therefore can be written as

$$r - a(1-e) = (M \sin k + N)^2, \tag{1}$$

where

$$M + N = (r' - a(1-e))^{1/2}; \qquad M - N = (r'' - a(1-e))^{1/2}. \tag{2}$$

The variable k is called the *inferior partial anomaly*.
For the segment of the orbit containing the apoapsis, we may consider the departure of $1/r$ from $r_{min} = 1/a(1+e)$. This departure should satisfy the following conditions.
1. To be a periodic function of one independent variable.
2. To be positive $\forall r$ between r' and r''.
3. To attain its maximum values at $1/r'$ and $1/r''$, and its minimum value at $1/a(1+e)$.
This departure therefore can be written as,

$$\frac{1}{r} - \frac{1}{a(1+e)} = (M' \sin k_1 + N')^2, \tag{3}$$

where

$$M' + N' = \left(\frac{1}{r'} - \frac{1}{a(1+e)}\right)^{1/2}; \ M' - N' = \left(\frac{1}{r''} - \frac{1}{a(1+e)}\right)^{1/2}. \quad (4)$$

The variable k_1 is called the *superior partial anomaly*.

By means of these departures, many findings are established for both k and k_1, namely: (i) A transformation relating the eccentric anomaly to k and a transformation relating the true anomaly to k_1. (ii) Expressions for defining each of k and k_1 in terms of the orbital elements. (iii) The interpretation and the intervals of definition of two moduli (X, S) related to k and k_1. (iv)The extreme values of the radius vector r and the elliptic equations in terms of k and k_1. (v) That for two radii vectors, r' and r'', the modulus X appearing in definition of the k and k_1 is a measure of the asymmetry of r' (or r'') from r'' (or r'), while the modulus S_{12} is a measure of the asymmetry of r' and r'' from the minimum value of r. (vi) A description of the segments represented by k and k_1. (vii) The relative position of the radius vector at $k = 0°$ and $k_1 = 180°$.

In what follows, we shall consider that the above equations are given and derive various conclusions associated with the segmentation, as well as provide additional interpretations to the parameters M, N, M' and N' appearing in these equations.

2. THE INFERIOR PARTIAL ANOMALY k

2.1. *The equation defining k*

This equation is,

$$r = a(1-e) + (M \sin k + N)^2, \quad (5)$$

where $M + N$ and $M - N$ are defined in equations (2), r' and r'' are any two radii vectors of the ellipse. The expression relating the radius vector and the eccentric anomaly for elliptic motion is:

$$r = a(1 - e\cos E) = a(1-e) + 2ae \sin^2 \frac{E}{2}. \quad (6)$$

Upon comparing equations (5) and (6) one obtains:

$$\sin\frac{E}{2} = \frac{1}{\sqrt{2ae}}\{M \sin k + N\}, \quad (7)$$

as the transformation relating the eccentric anomaly to the inferior partial anomaly k. Also, by equation (6) we can write equations (2) as:

$$M - N = \sqrt{2ae}\sin\frac{E''}{2}; \qquad M + N = \sqrt{2ae}\sin\frac{E'}{2}, \quad (8)$$

from which we obtain:

$$M = \sqrt{\frac{ae}{2}}\left\{\sin\frac{E'}{2} + \sin\frac{E''}{2}\right\}, \quad (9)$$

$$N = \sqrt{\frac{ae}{2}}\left\{\sin\frac{E'}{2} - \sin\frac{E''}{2}\right\}, \quad (10)$$

where E' and E'' are the eccentric anomalies corresponding to r' and r'' respectively. Further, setting

$$M = \frac{M}{\sqrt{2ae}} = S_{12}\cos X, \quad (11)$$

$$N = \frac{M}{\sqrt{2ae}} = S_{12}\sin X, \quad (12)$$

equations (5) and (7) could be written in terms of S_{12} and X as:

$$r = a(1-e) + 2aeS_{12}^2\left(\cos X \sin k + \sin X\right)^2, \quad (13)$$

$$\sin\frac{E}{2} = S_{12}\left(\cos X \sin k + \sin X\right). \quad (14)$$

Any one of the equations (5), (7), (13) or (14) could be used for the definition of the inferior partial anomaly k.

2.2. *The equations defining S_{12} and X in terms of r', r'' and in terms of E', E''*

From equations (11) and (12) we have:

$$S_{12} = \left(\frac{M^2+N^2}{2ae}\right)^{1/2}, \tag{15}$$

$$\tan(45-X) = \frac{M-N}{M+N}. \tag{16}$$

Using equations (2) we can write equations (15) and (16) in terms of r' and r'' as:

$$S_{12} = \left\{\frac{r'+r''-2a(1-e)}{4ae}\right\}^{1/2}, \tag{17}$$

$$\tan(45-X) = \left\{\frac{r''-a(1-e)}{r'-a(1-e)}\right\}^{1/2}. \tag{18}$$

Using equations (9) and (10) we can write equations (15) and (16) in terms of E' and E'' as:

$$S_{12} = \left\{\frac{\sin^2 E'/2+\sin^2 E''/2}{2}\right\}^{1/2}, \tag{19}$$

$$\tan(45-X) = \frac{\sin E''/2}{\sin E'/2}. \tag{20}$$

Equations (17) and (18) are the required equations defining S_{12} and X in terms of r' and r'', while equations (19) and (20) are the corresponding equations in terms of E' and E''.

2.3. *The intervals of definition for S_{12} and X*

Since for any radius vector r of the ellipse we have:

$$\max(r) = a(1+e);\ \min(r) = a(1-e) \tag{21}$$

Consequently, for the two radii vectors r' and r'' we have:

$$\max\left[r'-a(1-e)\right] = \max\left[r''-a(1-e)\right] = 2ae, \tag{22}$$

$$\min\left[r'-a(1-e)\right] = \min\left[r''-a(1-e)\right] = 0, \tag{23}$$

$$\max\left[r'+r''\right] = \max(r')+\max(r'') = 2a(1+e), \tag{24}$$

$$\min\left[r'+r''\right] = \min(r')+\min(r'') = 2a(1-e). \tag{25}$$

Equation (17) could be written as:

$$\min\left\{\frac{r'+r''-2a(1-e)}{4ae}\right\}^{1/2} \le S_{12} \le \max\left\{\frac{r'+r''-2a(1-e)}{4ae}\right\}^{1/2}. \tag{26}$$

Then, by equations (24) and (25) this inequality becomes:

$$0 \le S_{12} \le 1. \tag{27}$$

In addition, we can write equation (18) as:

$$\min\left\{\frac{r''-a(1-e)}{r'-a(1-e)}\right\}^{1/2} \le \tan(45-X) \le \max\left\{\frac{r''-a(1-e)}{r'-a(1-e)}\right\}^{1/2}, \tag{28}$$

and then

$$\frac{\min(r'' - a(1-e))^{1/2}}{\max(r' - a(1-e))^{1/2}} \leq \tan(45 - X) \leq \frac{\max(r'' - a(1-e))^{1/2}}{\min(r' - a(1-e))^{1/2}}. \tag{29}$$

Using equations (22) and (23), the inequality (29) becomes:

$$0 \leq \tan(45 - X) \leq \infty, \tag{30}$$

and then we have:

$$-\frac{\pi}{4} \leq X \leq \frac{\pi}{4}. \tag{31}$$

Inequalities (27) and (31) are what we need to obtain.

2.4. *The extreme values of the radius vector r and the elliptic equations in terms of k*

Equation (13) could be written as:

$$\begin{aligned} r &= a\left\{1 - e + eS_{12}^2 + eS_{12}^2 \sin^2 X\right\} + 2aeS_{12}^2 \sin 2X \sin k \\ &\quad -aeS_{12}^2 \cos^2 X \cos 2k. \end{aligned} \tag{32}$$

Consequently,

$$\frac{dr}{dk} = 2aeS_{12}^2 \left\{\sin 2X \cos k + \cos^2 X \sin 2k\right\}. \tag{33}$$

The necessary condition for the extreme values of r is $dr/dk = 0$,

$$\begin{array}{c} \text{that is} \\ \text{or} \end{array} \left\{ \begin{array}{l} \sin 2X \cos k + \cos^2 X \sin 2k = 0, \\ \cos X \cos k\{\sin X + \sin k \cos X\} = 0. \end{array} \right. \tag{34}$$

Therefore, the extreme values of r when expressed in terms of k occur at

$$k = 90°, \quad k = 270°, \quad \sin k = -\tan X. \tag{35}$$

Differentiating equation (33) with respect to k, we obtain:

$$\frac{d^2r}{dk^2} = 2aeS_{12}^2\{-\sin 2X \sin k + 2\cos^2 X \cos 2k\}. \tag{36}$$

Let us test the values of k given in equations (35) for the extreme values of r.

At $k = 90°$:
For this value of k, equation (36) becomes:

$$\frac{d^2r}{dk^2} = -2aeS_{12}^2\{\sin 2X + 2\cos^2 X\}. \tag{37}$$

Since $-\pi/4 \leq X \leq \pi/4$, it follows that:

$$\sin 2X + 2\cos^2 X \geq 0. \tag{38}$$

Consequently,

$$\frac{d^2r}{dk^2} \mid_{k=90°} \leq 0. \tag{39}$$

That is to say, at $k = 90° \ \forall -\pi/4 \leq X \leq \pi/4$, r is maximum. Let this maximum be r_1. By equation (32) for $k = 90°$, we get for r_1 the value:

$$r_1 = a(1-e) + 2aeS_{12}^2 + 2aeS_{12}^2 \sin 2X. \tag{40}$$

From equations (11) and (12) we obtain:

$$S_{12}^2 \sin 2X = \frac{MN}{ae}. \tag{41}$$

Using equations (9) and (10) in this equation yields:

$$S_{12}^2 \sin 2X = \frac{1}{2}\left\{\sin \frac{E'}{2}^2 - \sin \frac{E''}{2}^2\right\}, \tag{42}$$

or

$$S_{12}^2 \sin 2X = \frac{1}{2}\left\{\frac{r' - r''}{4ae}\right\}. \tag{43}$$

Using equations (43) and (17) in equation (40) it gives:

$$r_1 = r'. \tag{44}$$

This equation could be obtained from equation (5) for $k = 90°$, by comparing the resulting equation with equations (2).

At $k = 270°$:
For this value of k, equation (36) becomes:

$$\frac{d^2r}{dk^2} = 2aeS_{12}^2\{\sin 2X - 2\cos^2 X\}. \tag{45}$$

Again, since $-\pi/4 \leq X \leq \pi/4$, it follows that:

$$\sin 2X - 2\cos^2 X \leq 0. \tag{46}$$

Consequently,

$$\frac{d^2r}{dk^2}\,|_{k=270°} \leq 0. \tag{47}$$

That is to say, at $k = 270°\ \forall - \pi/4 \leq X \leq \pi/4$, r is maximum. Let this maximum be r_2. By equation (32) for $k = 270°$ we get:

$$r_2 = a(1-e) + 2aeS_{12}^2 - 2aeS_{12}^2 \sin 2X. \tag{48}$$

Using equations (43) and (17) in equation (48) yields:

$$r_2 = r''. \tag{49}$$

This equation could be obtained from equation (5) for $k = 270°$, by comparing the resulting equation with equations (2).

At $\sin k = -\tan X = -N/M$:
Equation (5) with $\sin k = -N/M$ or equation (32) with $\sin k = -\tan X$ will give:

$$r = a(1-e). \tag{50}$$

Therefore, at $\sin k = -\tan X$ r is minimum.
From the above analysis we have the following results.

Result 1:
The periodic representation of the radius vector r in terms of the inferior partial anomaly k [equation (5) or equation (13)] has two maxima, r'' and r', when $k = 270°$ and $k = 90°$ respectively, and has a minimum $a(1-e)$ when $\sin k = -N/M$ or $\sin k = -\tan X$.

Now we are in a position to obtain the elliptic equations in terms of k, and this is done as follows. We have already obtained the expression of r in terms of k as given in equation (32).
Since

$$r\sin f = a(1-e^2)^{1/2}\sin E = 2a(1-e^2)^{1/2}\sin\frac{E}{2}\cos\frac{E}{2}, \tag{51}$$

and

$$\sin\frac{E}{2} = S_{12}\left(\cos X\sin k + \sin X\right), \tag{52}$$

we can write

$$r\sin f = 2aS_{12}(1-e^2)^{1/2}\left(\cos X\sin k + \sin X\right)A, \tag{53}$$

where

$$A = \left(1 - S_{12}^2\sin^2 X - S_{12}^2\cos^2 X\sin^2 k - S_{12}^2\sin 2X\sin k\right)^{1/2} = \cos\frac{E}{2}. \tag{54}$$

Also, since

$$r\cos f = a(\cos E - e) = a\left(1 - e - 2\sin^2\frac{E}{2}\right), \tag{55}$$

using the expression of $\sin E/2$ in terms of k we obtain:

$$r\cos f = a\left\{1 - e - 2S_{12}^2\left(\cos^2 X\sin^2 k + \sin 2X\sin k + \sin^2 X\right)\right\}, \tag{56}$$

that is,

$$r\cos f = a\left(1 - e - S_{12}^2 - S_{12}^2\sin^2 X\right) - 2aS_{12}^2\sin 2X\sin k + aS_{12}^2\cos^2 X\cos 2k. \tag{57}$$

By equation (14) we have:

$$\frac{1}{2}\cos\frac{E}{2}dE = S_{12}\cos X\cos k\; dk, \tag{58}$$

or

$$dE = \frac{2S_{12}\cos X\cos k}{A}dk \tag{59}$$

where A is given by equation (54).
Since

$$n_c dt = \left(\frac{r}{a}\right)dE, \tag{60}$$

then

$$n_c dt = \left(\frac{r}{a}\right)\left\{\frac{2S_{12}\cos X\cos k}{A}\right\}dk. \tag{61}$$

where n_c is the mean motion. Equations (51) to (61) in addition to equation (32) are the required elliptic equations in terms of k.

2.5. *Some remarks concerning the inferior partial anomaly k*

It is evidently shown by the above analysis that certain points need discussion. In the following, some important remarks are given.

2.5.1. *Interpretation of X and S_{12}*

From equation (32) we have;

$$r'' = a\left\{1 - e + eS_{12}^2 + eS_{12}^2\sin^2 X\right\} - 2aeS_{12}^2\sin 2X + aeS_{12}^2\cos^2 X. \tag{62}$$

and

$$r' = a\left\{1 - e + eS_{12}^2 + eS_{12}^2\sin^2 X\right\} + 2aeS_{12}^2\sin 2X + aeS_{12}^2\cos^2 X. \tag{63}$$

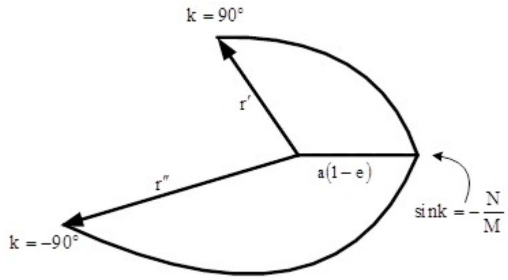

Fig. 2. Description of the segment represented by the inferior partial anomaly k.

By these equations, and equations (18), (31) we have,

$$\begin{cases} r' > r'' \Leftrightarrow X \epsilon [0, \pi/4], \\ r' < r'' \Leftrightarrow X \epsilon [-\pi/4, 0], \\ r' = r'' \neq a(1-e) \Leftrightarrow X = 0, \\ r'' = a(1-e) \neq r' \Leftrightarrow X = \pi/4, \\ r' = a(1-e) \neq r'' \Leftrightarrow X = -\pi/4. \end{cases} \quad (64)$$

By these equations, and the equations defining S_{12} , the interpretation of X and S_{12} may be as follows.

Result 2:
The modulus X appearing in the definition of the inferior partial anomaly k is a measure of the asymmetry of r'(or r'') from r''(or r'), while the modulus S_{12} is a measure of the asymmetry of r' and r'' from the minimum value of r, which is $a(1-e)$.

2.5.2. *Description of the segment represented by k*

Equations (51) to (61), in addition to equation (32), show the following

- At $r = r'', k = 270°$. As r decreases, k increases.
- After passing the periapsis, r increases as does k, until at $r = r'$ we have $k = 90°$.
- If we allow k to increase beyond 90° we retrace the same segment of the ellipse in reverse order.

Thus, we can describe the segment of the ellipse represented by the inferior partial anomaly k as follows.

Result 3:
The inferior partial anomaly k represents the segment of the ellipse from the periapsis to $r = r'$ on one side of the major axis, where $0° \leq E \leq 180°$, and from the periapsis to $r = r''$ on the other side of the major axis, where $180° \leq E \leq 360°$.
As k is varied from 0° to 360°, equations (5) or (13) and equations (7) or (14) give the coordinates of the ellipse, r and E , only in the segment formed by r' and r'' as indicated in Figure 2 (in which $r'' > r'$).

2.5.3. *Relative position of the radius vector at $k = 0$*

Let the value of r at $k = 0$ be r_0; then, by equation (32) we have

$$r_0 = a(1-e) + 2aeS_{12}^2 \sin^2 X. \quad (65)$$

From equations (62), (63) and (65) we have;

$$r'' - r_0 = 2aeS_{12}^2 \cos X(\cos X - 2\sin X), \tag{66}$$

and

$$r' - r_0 = 2aeS_{12}^2 \cos X(\cos X + 2\sin X). \tag{67}$$

Now we have to consider the following cases:

1. If $r' > r''$.
According to the first relation in (64), we have $\cos X < 2\sin X, \sin X > 0$ and $\cos X > 0$. Hence, by these conditions and equations (66) and (67) we have for this case;

$$r_0 > r''; \qquad r_0 < r'. \tag{68}$$

2. If $r' < r''$.
According to the second relation in (64), we have $\cos X > 0$ and $\sin X < 0$. Hence, by these conditions and equations (66) and (67) we have for this case;

$$r_0 < r''; \qquad r_0 > r'. \tag{69}$$

3. If $r' = r'' \neq a(1-e)$.
According to the third relation in (64) and equation (65) we have for this case;

$$r_0 = a(1-e). \tag{70}$$

From equations (68), (69) and (70), we may conclude the following result.

Result 4:
For $r' < r''$ or $r' > r''$, the radius vector r_0 corresponding to $k = 0$ lies on the same side of the major axis as the $\max\{r', r''\}$, while for $r' = r'' \neq a(1-e)$, r_0 occurs at the periapsis. Figure 3 illustrates this result.

This section completes the analysis of the inferior partial anomaly k. In the following section we shall consider the superior partial anomaly k_1.

3. THE SUPERIOR PARTIAL ANOMALY k_1

3.1. *The equation defining k_1*

This equation is:

$$\frac{1}{r} = \frac{1}{a(1+e)}(M' \sin k_1 + N')^2, \tag{71}$$

where

$$M' - N' = \left\{\frac{1}{r''} - \frac{1}{a(1+e)}\right\}^{1/2}, \tag{72}$$

$$M' + N' = \left\{\frac{1}{r'} - \frac{1}{a(1+e)}\right\}^{1/2}. \tag{73}$$

The expression relating the radius vector to the true anomaly f for elliptic motion is:

$$\frac{1}{r} = \frac{1 + e\cos f}{a(1-e)} = \frac{(1-e) + 2e\cos^2(f/2)}{a(1-e^2)} = \left\{\frac{1}{a(1+e)} + \frac{2e\cos^2(f/2)}{a(1-e^2)}\right\}. \tag{74}$$

Upon comparing (71) and (74) one obtains:

$$\cos\frac{f}{2} = \sqrt{\frac{a(1-e^2)}{2e}}\,(M' \sin k_1 + N'), \tag{75}$$

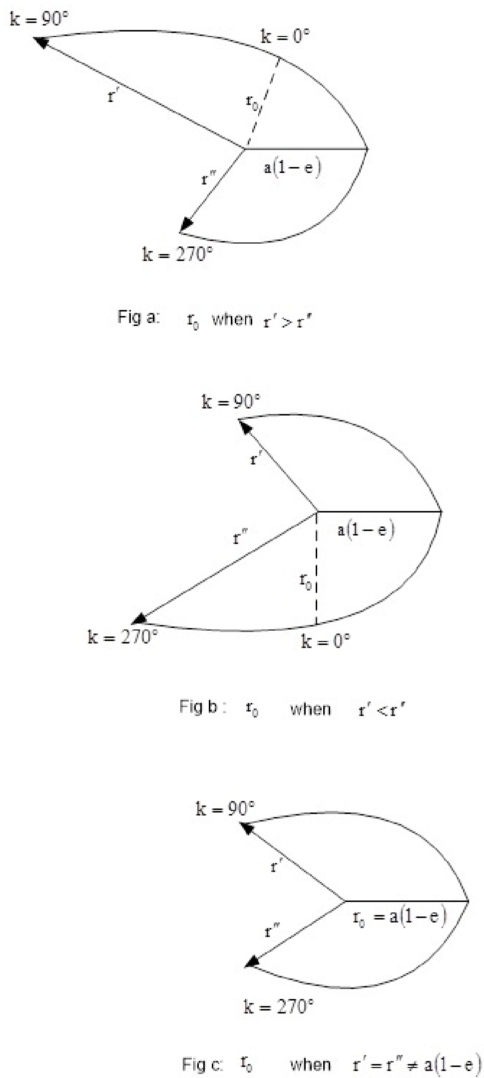

Fig. 3. Relative position of the radius vector r_0 for the three cases: $r' < r'', r' > r''$ and $r' = r'' \neq a(1-e)$.

as the transformation relating the true anomaly to the superior partial anomaly k_1.
From equation (74) we have:

$$\cos\frac{f}{2} = \pm\sqrt{\frac{a(1-e^2)}{2e}}\left\{\frac{1}{r} - \frac{1}{a(1+e)}\right\}^{1/2}. \tag{76}$$

From the analysis of § 2.5 we have $f' \leq 180°$ and $f'' \geq 180°$, then we must have:

$$\cos\frac{f'}{2} = \sqrt{\frac{a(1-e^2)}{2e}}\left\{\frac{1}{r'} - \frac{1}{a(1+e)}\right\}^{1/2}, \tag{77}$$

and

$$\cos\frac{f''}{2} = -\sqrt{\frac{a(1-e^2)}{2e}}\left\{\frac{1}{r''} - \frac{1}{a(1+e)}\right\}^{1/2}. \tag{78}$$

Hence, by these equations, equations (72) and (73) could be written as:

$$N' - M' = \sqrt{\frac{2e}{a(1-e^2)}}\cos\frac{f''}{2};\ N' + M' = \sqrt{\frac{2e}{a(1-e^2)}}\cos\frac{f'}{2}, \tag{79}$$

hence

$$N' = \sqrt{\frac{2e}{a(1-e^2)}}\left(\cos\frac{f'}{2} + \cos\frac{f''}{2}\right)/2, \tag{80}$$

and

$$M' = \sqrt{\frac{2e}{a(1-e^2)}}\left(\cos\frac{f'}{2} - \cos\frac{f''}{2}\right)/2. \tag{81}$$

Let

$$M'\sqrt{\frac{a(1-e^2)}{2e}} = S'_{22}\cos X', \tag{82}$$

$$N'\sqrt{\frac{a(1-e^2)}{2e}} = -S'_{22}\sin X'. \tag{83}$$

By means of equations (82) and (83) we can write equations (71) and (75) in terms of S'_{22} and X' as:

$$\frac{1}{r} = \frac{1}{a(1+e)} + \frac{2eS'^2_{22}}{a(1-e^2)}\left(\cos X'\sin k_1 - \sin X'\right)^2, \tag{84}$$

$$\cos\frac{f}{2} = S'_{22}\left(\cos X'\sin k_1 - \sin X'\right). \tag{85}$$

Any of the equations (71), (75), (84) or (85) may be used for the definition of the superior partial anomaly k_1.

3.2. *Equations defining S'_{22} and X' in terms of r', r'', and in terms of f', f''*

From equations (82) and (83) we have:

$$S'_{22} = \sqrt{\frac{a(1-e^2)}{2e}}(M'^2 + N'^2)^{1/2}, \tag{86}$$

$$\tan(45 - X') = \frac{M' + N'}{M' - N'}. \tag{87}$$

Using equations (80), (81) we can write equations (86), (87) in terms of f' and f'' as:

$$S'^2_{22} = \left\{\left(\cos^2 f'/2 + \cos^2 f''/2\right)/2\right\}^{1/2}, \tag{88}$$

$$\tan(45 - X') = -\frac{\cos f'/2}{\cos f''/2}, \tag{89}$$

where f' and f'' are the true anomalies at r' and r'' respectively.
Using equations (72), (73) we can write equations (86), (87) in terms of r' and r'' as:

$$S'^2_{22} = \left\{\frac{a(r'+r'')(1+e) - 2r'r''}{4er'r''}(1-e)\right\}^{1/2}, \tag{90}$$

$$\tan(45 - X') = \left\{\frac{a(1+e) - r'}{a(1+e) - r''}\frac{r''}{r'}\right\}^{1/2}. \tag{91}$$

3.3. *The intervals of definition for S'_{22} and X'*

For any radius vector r of the ellipse we have:

$$\max(r) = a(1+e); \qquad \min(r) = a(1-e), \tag{92}$$

consequently

$$\max\left(\frac{1}{r}\right) = \frac{1}{a(1-e)}; \qquad \min\left(\frac{1}{r}\right) = \frac{1}{a(1+e)}, \tag{93}$$

and

$$\max\{a(1+e) - r\} = 2ae; \qquad \min\{a(1+e) - r\} = 0. \tag{94}$$

Also for any radius vectors r' and r'' we have:

$$\max\left(\frac{1}{r'} + \frac{1}{r''}\right) = \frac{2}{a(1-e)}; \qquad \min\left(\frac{1}{r'} + \frac{1}{r''}\right) = \frac{2}{a(1+e)}. \tag{95}$$

Equation (90) could be written as:

$$\min(U) \leq S'_{22} \leq \max(U), \tag{96}$$

where

$$U = \left(\frac{1-e}{4e}\right)^{1/2} \left\{a(1+e)\left(\frac{1}{r'} + \frac{1}{r''}\right) - 2\right\}^{1/2}. \tag{97}$$

Then by equation (95) the inequality (96) becomes:

$$0 \leq S'_{22} \leq 1. \tag{98}$$

In addition, we can write equation (91) as:

$$\min\left\{\frac{a(1+e) - r'}{a(1-e) - r''}\frac{r''}{r'}\right\}^{1/2} \leq \tan(45 - X') \leq \max\left\{\frac{a(1+e) - r'}{a(1-e) - r''}\frac{r''}{r'}\right\}^{1/2}, \tag{99}$$

or

$$\min(V_1) \leq \tan(45 - X') \leq \max(V_2), \tag{100}$$

where

$$V_1 = \frac{\min(a(1+e) - r')^{1/2}.\min(r'')^{1/2}}{\max(a(1+e) - r'')^{1/2}.\max(r')^{1/2}}, \tag{101}$$

and

$$V_2 = \frac{\max(a(1+e) - r')^{1/2}.\max(r'')^{1/2}}{\min(a(1+e) - r'')^{1/2}.\min(r')^{1/2}}. \tag{102}$$

By equation (94) this inequality becomes:

$$0 \leq \tan(45 - X') \leq \infty, \tag{103}$$

which gives

$$-\pi/4 \leq X' \leq \pi/4. \tag{104}$$

The inequalities (98) and (104) are what we need to obtain.

3.4. *The extreme values of the radius vector r and the elliptic equations in terms of k_1*

Equation (84) could be written as:

$$\frac{a(1-e^2)}{r} = 1 - e + eS_{22}'^2 + eS_{22}'^2 \sin^2 X' - 2eS_{22}'^2 \sin 2X' \sin k_1 - eS_{22}'^2 \cos^2 X' \cos 2k_1. \tag{105}$$

Differentiating equation (105) with respect to k_1 we obtain:

$$a(1-e^2)\frac{dr}{dk_1} = 2eS_{22}'^2 r^2 \left\{\sin 2X' \cos k_1 - \cos^2 X' \sin 2k_1\right\}. \tag{106}$$

Since the necessary condition for the extreme values of r is $dr/dk_1 = 0$, then

$$\cos X' \cos k_1 (\sin X' - \sin k_1 \cos X') = 0. \tag{107}$$

Therefore, the extreme values of r when expressed in terms of k_1 occur at:

$$k_1 = 90°;\ k_1 = 270°;\ \sin k_1 = \tan X'. \tag{108}$$

Differentiating equation (106) with respect to k_1 we obtain:

$$a(1-e^2)\frac{d^2r}{dk_1^2} = 2eS_{22}'^2 \left\{ r^2 \left(-\sin 2X' \sin k_1 - 2\cos^2 X' \cos 2k_1\right) + \left(\sin 2X' \cos k_1 - \cos^2 X' \sin 2k_1\right)\left(2r\frac{dr}{dk_1}\right)\right\}. \tag{109}$$

Now we shall test the values of k_1 given in equation (108) for the extreme values of r.

At $k_1 = 90°$:
For this value of k_1, $dr/dk_1 = 0$ and hence equation (109) gives:

$$\frac{d^2r}{dk_1^2}\,|_{k_1=90°} = \left[\frac{-2eS_{22}'^2}{a(1-e^2)}\right](r^2)_{k_1=90°}\left\{\sin 2X' - 2\cos^2 X'\right\}. \tag{110}$$

Since $-\pi/4 \leq X' \leq \pi/4$, it follows that:

$$\sin 2X' - 2\cos^2 X' \leq 0. \tag{111}$$

By this condition, and the fact that $2eS_{22}'^2 (r^2)_{k_1=90°}/a(1-e^2) > 0$, it follows that:

$$\frac{d^2r}{dk_1^2}_{k_1=90°} \geq 0. \tag{112}$$

That is to say, at $k_1 = 90°$, $\forall -\pi/4 \leq X' \leq \pi/4$, r is minimum. Let this minimum be r_1'. By equation (105) for $k_1 = 90°$ we get:

$$\frac{a(1-e^2)}{r_1'} = 1 - e + eS_{22}'^2 - 2eS_{22}'^2 \sin 2X'. \tag{113}$$

From equations (82) and (83) we get:

$$S_{22}'^2 \sin 2X' = -M'N'\left[\frac{a(1-e^2)}{e}\right]. \tag{114}$$

Using equations (80) and (81) in this equation gives:

$$S_{22}'^2 \sin 2X' = \left\{\frac{\cos^2 f''/2 - \cos^2 f'/2}{2}\right\}, \tag{115}$$

or

$$S_{22}'^2 \sin 2X' = \left\{\frac{a(1-e^2)(r'-r'')}{4er'r''}\right\}. \tag{116}$$

Using equations (116) and (90) in equation (113) gives:

$$\frac{a(1-e^2)}{r'_1} = 1 - e + 2e\left\{\frac{a(r'+r'')(1-e^2) - 2r'r''(1-e)}{4er'r''}\right\} + 2e\left\{\frac{a(1-e^2)(r''-r')}{4er'r''}\right\}, \tag{117}$$

or

$$\frac{a(1-e^2)}{r'_1} = 1 - e + \frac{a(1-e^2)}{r'} - (1-e). \tag{118}$$

Then

$$r'_1 = r'. \tag{119}$$

This equation could be found from equation (71) with $k_1 = 90°$, by comparing the resulting equation with equation (73).

At $k_1 = 270°$:
Since for this value $dr/dk_1 = 0$, equation (109) gives:

$$\frac{d^2r}{dk_1^2}\,|_{k_1=270°} = \left[\frac{2eS'^2_{22}}{a(1-e^2)}\right](r^2)_{k_1=270°}\left\{\sin 2X' + 2\cos^2 X'\right\}. \tag{120}$$

Again, since $-\pi/4 \leq X' \leq \pi/4$, it follows that:

$$\sin 2X' + 2\cos^2 X' \geq 0. \tag{121}$$

From this condition, and from the fact that $2eS'^2_{22}(r^2)_{k_1=270°}/a(1-e^2) > 0$, it follows that:

$$\frac{d^2r}{dk_1^2}_{k_1=270°} \geq 0. \tag{122}$$

That is to say, at $k_1 = 270°\ \forall -\pi/4 \leq X' \leq \pi/4$, r is minimum. Let this minimum be r'_2. By equation (105) for $k_1 = 270°$ we obtain:

$$\frac{a(1-e^2)}{r'_2} = 1 - e + 2eS'^2_{22} + 2eS'^2_{22}\sin 2X'. \tag{123}$$

Using equations (116) and (90) in this equation, gives:

$$r'_2 = r''. \tag{124}$$

This equation could be found from equation (71) with $k_1 = 270°$, by comparing the resulting equation with equation (72).

At $\sin k_1 = -N'/M' = \tan X'$:
Equation (71) with $\sin k_1 = -N'/M'$ or equation (105) with $\sin k_1 = \tan X'$ will give:

$$r = a(1+e). \tag{125}$$

Therefore, at $\sin k_1 = -N'/M' = \tan X'$, r is maximum. From the above analysis we can summarize

Result 5
The periodic representation of the radius vector r in terms of the superior partial anomaly k_1 [equation (71) or equation (105)] has two minima, r'' and r', when $k_1 = 270°$ and $k_1 = 90°$ respectively, and has a maximum $a(1+e)$, when $\sin k_1 = -N'/M'$ or $(\sin k_1 = \tan X')$.

Now we are in a position to obtain the elliptic equations in terms of k_1. This is done as follows.

We have already obtained the expression of r in terms of k_1 as:

$$\frac{1}{r} = D/a(1-e^2), \tag{126}$$

where

$$D = (1 - e + eS_{22}'^2 + eS_{22}'^2 \sin^2 X') - 2eS_{22}'^2 \sin 2X' \sin k_1 - eS_{22}'^2 \cos^2 X' \cos 2k_1. \quad (127)$$

Since

$$\cos f = 2\cos^2 f/2 - 1, \quad (128)$$

then by using equation (85) we get $\cos f$ in terms of k_1 as:

$$\cos f = (S_{22}'^2 + S_{22}'^2 \sin^2 X' - 1) - 2S_{22}'^2 \sin 2X' \sin k_1 - S_{22}'^2 \cos^2 X' \cos 2k_1. \quad (129)$$

Also, since:

$$\sin f = 2\sin f/2 \cos f/2 = 2\cos f/2 \left(1 - \cos^2 f/2\right)^{1/2} = 2C \cos f/2, \quad (130)$$

where

$$C = \left\{1 - S_{22}'^2 \sin^2 X' + S_{22}'^2 \sin 2X' \sin k_1 - S_{22}'^2 \cos^2 X' \sin^2 k_1\right\}^{1/2}, \quad (131)$$

then by equation (85) we get $\sin f$ in terms of k_1 as:

$$\sin f = 2S_{22}'^2(\cos X' \sin k_1 - \sin X')C. \quad (132)$$

Since

$$r = a(1 - e\cos E) = a(1 - e^2)(1 + e\cos f)^{-1}, \quad (133)$$

then we can write:

$$\sin E \; dE = \frac{(1 - e^2)\sin f}{(1 + e\cos f)^2} \; df = \frac{r^2(1 - e^2)\sin f}{a^2(1 - e^2)^2} \; df = \frac{r^2 \sin f}{a^2(1 - e^2)} \; df. \quad (134)$$

Since

$$r \sin f = a(1 - e^2)^{1/2} \sin E \Rightarrow \sin E = \frac{r \sin f}{a(1 - e^2)^{1/2}}. \quad (135)$$

Hence, by using equation (135) in the left hand side of equation (134) we obtain:

$$dE = \left(\frac{r}{a}\right)(1 - e^2)^{-1/2} \; df. \quad (136)$$

Therefore, $n_c dt$ in terms of f is written as

$$n_c dt = \left(\frac{r}{a}\right)^2 (1 - e^2)^{-1/2} \; df. \quad (137)$$

Again, by equation (85) we have;

$$df = -2S_{22}' \cos X' \cos k_1/C \; dk_1, \quad (138)$$

where C is given by (131).

Using equation (138) in equation (137) yields for $n_c dt$ in terms of k_1 the formula;

$$n_c dt = -2\left(\frac{r}{a}\right)^2 (1 - e^2)^{-1/2} S_{22}' \cos X' \cos k_1/C \; dk_1. \quad (139)$$

Equations (126), (129), (132) and (139) are the required elliptic equations in terms of k_1.

3.5. *Some remarks concerning the superior partial anomaly* k_1

Corresponding to § 3.4, the following remarks are given for the superior partial anomaly k_1.

3.5.1. *The interpretation of* X' *and* S'_{22}

From equation (126) we have;

$$\frac{a(1-e^2)}{r'} = 1 - e + 2eS'^2_{22} - 2eS'^2_{22}\sin 2X'. \tag{140}$$

and

$$\frac{a(1-e^2)}{r''} = 1 - e + 2eS'^2_{22} + 2eS'^2_{22}\sin 2X'. \tag{141}$$

Using these equations, and equations (91), (104) we get;

$$\begin{cases} r' > r'' \Leftrightarrow X' \epsilon [0, \pi/4], \\ r' < r'' \Leftrightarrow X' \epsilon [-\pi/4, 0], \\ r' = r'' \neq a(1+e) \Leftrightarrow X' = 0, \\ r' = a(1+e) \Leftrightarrow X' = \pi/4, \\ r'' = a(1+e) \Leftrightarrow X' = -\pi/4. \end{cases} \tag{142}$$

By these equations, and the equations defining S'_{22} , the interpretation of X' and S'_{22} may be as follows:

Result 6:
The modulus X' appearing in the definition of the superior partial anomaly k_1 is a measure of the asymmetry of r'(or r'') from r''(or r'), while the modulus S'_{22} is a measure of the asymmetry of r' and r'' from the maximum value of r, which is $a(1+e)$.

3.5.2. *Description of the segment represented by* k_1

The elliptic equations in terms of k_1 show that:

- At $r = r'$, we have $k_1 = 90°$. As r increases, so does k_1.
- After passing the apoapsis, r decreases, k_1 increases until at $r = r''$ we have $k_1 = 270°$.
- If k_1 increases beyond $270°$, the same segment of the ellipse is retraced in the reverse order.

From these notes we can describe the segment of the ellipse represented by the superior partial anomaly k_1 as follows.

Result 7:
The superior partial anomaly k_1 represents the segment of the ellipse from the apoapsis to $r = r'$ on the side of the major axis where $0° \leq E \leq 180°$, and from apoapsis to $r = r''$ on the other side of the major axis where $180° \leq E \leq 360°$. As k_1 is varied from $0°$ to $360°$, equations (71) or (105) and equations (75) or (85) give the coordinates of the ellipse , r and f, only in the segment formed by r' and r'', as indicated in Figure 4 (in which $r'' > r'$).

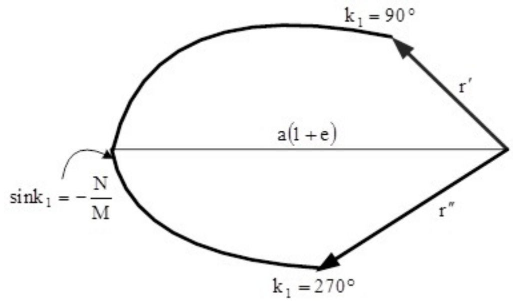

Fig. 4. Description of the segment represented by the superior partial anomaly k_1.

3.5.3. *Relative position of the radius vector at* $k_1 = 180^0$

Let the value of r at $k_1 = 180°$ be r'_0; then, by equation (126) we have;

$$\frac{a(1-e^2)}{r'_0} = 1 - e + 2eS'^2_{22}\sin^2 2X'. \tag{143}$$

From equations (140), (141) and (143) we have

$$a(1-e^2)\left(\frac{1}{r'} - \frac{1}{r'_0}\right) = 2eS'^2_{22}\cos X'\left[\cos X' - 2\sin X'\right], \tag{144}$$

or

$$a(1-e^2)\left(\frac{1}{r''} - \frac{1}{r'_0}\right) = 2eS'^2_{22}\cos X'\left[\cos X' + 2\sin X'\right]. \tag{145}$$

Now we shall consider the following cases.

1. If $r' > r''$.
According to the first relation in (142), we have $\cos X' < 2\sin X'$, $\sin X' > 0$ and $\cos X' > 0$. Hence, by these conditions and equations (144) and (145) we have for this case

$$r'_0 > r''; \; r'_0 < r'. \tag{146}$$

2. If $r' < r''$.
According to the second relation in (142), we have $\cos X' > 0$ and $\sin X' < 0$. Hence, by these conditions and equations (144) and (145) we have for this case

$$r'_0 < r''; \; r'_0 > r'. \tag{147}$$

3. If $r' = r'' \neq a(1+e)$.
According to the third relation in (142) and equation (143)we have for this case

$$r'_0 = a(1+e). \tag{148}$$

From equations (145), (146) and (147), we may state the following result.

Result 8:
For $r' > r''$ or $r' < r''$, the radius vector r'_0 corresponding to $k_1 = 180°$ lies on the same side of the major axis as the $\min\{r', r''\}$, while for $r' = r'' \neq a(1+e)$, r'_0 occurs at the apoapsis. Figure 5 illustrates this result. This section completes the analysis of the superior partial anomaly k_1. This analysis leads to the following conclusions.

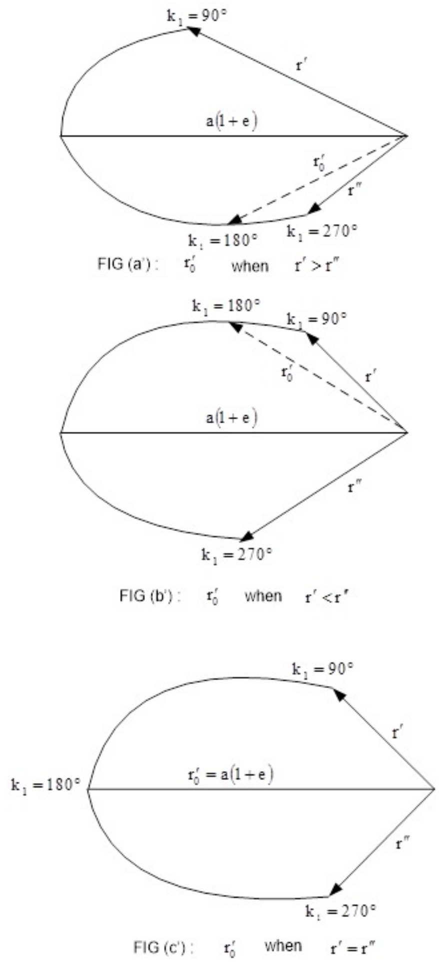

Fig. 5. Relative position of the radius vector r'_0 for the three cases: $r' > r'', r' < r''$ and $r' = r'' \neq a(1+e)$.

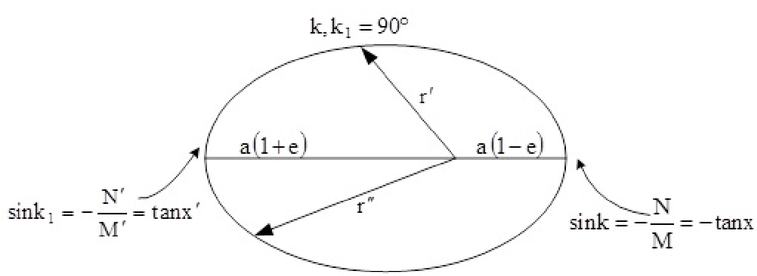

Fig. 6. Division the ellipse into two segments.

4. CONCLUSIONS

In the present paper, a novel analysis is given to show how Hansens inferior and superior partial anomalies k and k_1 can be used to divide the elliptic orbit into two segments (see Figure 6). The first segment includes the periapsis and is represented by the inferior partial anomaly k. The second segment includes the apoapsis and is represented by the superior partial anomaly k_1. The main findings of this manuscript can be summarized as follows.

- The periodic representation of the radius vector r in terms of the inferior partial anomaly k has two maxima r'' and r' when $k = 270°$ and $k = 90°$ respectively, and a minimum $a(1-e)$ when $\sin k = -\tan X$.
- The inferior partial anomaly k represents the segment of the ellipse from the periapsis to $r = r'$ on one side of the major axis where $0° \leq E \leq 180°$ and from the periapsis to $r = r''$ on the other side of the major axis, where $180° \leq E \leq 360°$.
- For $r' < r''$ or $r' > r''$ the radius vector r_0 corresponding to $k = 0$ lies on the same side of the major axis as the $\max\{r', r''\}$, while for $r' = r'' \neq a(1-e)$, r_0 occurs at the periapsis.
- The periodic representation of the radius vector r in terms of the superior partial anomaly k_1 has two minima r'' and r' when $k_1 = 270°$ and $k_1 = 90°$ respectively, and has a maximum $a(1+e)$ when $\sin k = -\tan X$.
- The superior partial anomaly k_1 represents the segment of the ellipse from the apoapsis to $r = r'$ on one side of the major axis where $0° \leq E \leq 180°$ and from the apoapsis to $r = r''$ on the other side of the major axis, where $180° \leq E \leq 360°$.
- For $r' > r''$ or $r' < r''$ the radius vector r'_0 corresponding to $k_1 = 180°$ lies on the same side of the major axis as the $\min\{r', r''\}$, while for $r' = r'' \neq a(1+e)$, r'_0 occurs at the apoapsis.

The authors are grateful to the referee for valuable comments that improved the original manuscript.

A. APPENDIX

Table of symbols

a: The semi-major axis of elliptic orbit	e: The eccentricity of elliptic orbit
f: The true anomaly [rad]	E: The eccentric anomaly [rad]
n_c:The mean motion [rad/s]	t: The time [s]
k: The inferior partial anomaly by Hansen	k_1: The superior partial anomaly by Hansen
r: The rdius vector	

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