RevMexAA (Serie de Conferencias), **21**, 152–155 (2004)

THE MODIFIED RESTRICTED THREE BODY PROBLEMS

Ing-Guey Jiang[1] and Li-Chin Yeh[2]

RESUMEN

El problema restringido de los tres cuerpos es importante en la dinámica de las estrellas dobles y múltiples y de los sistemas planetarios. Extendemos la versión clásica de este problema a una situación que incluye un anillo. Encontramos puntos de equilibrio y curvas muy distintas a las del caso clásico. Calculamos el valor del exponente de Lyapunov para algunas órbitas.

ABSTRACT

The restricted three body problem is well-known and very important for the dynamics of binary and multiple stars and also planetary systems. We extend the classical version of this problem to the situation that there are some external forces from the belt. We find that both the equilibrium points and solution curves become quite different from the classical case. We also determine the values of Lyapunov exponent for some important orbits.

Key Words: **THREE-BODY PROBLEM — STELLAR DYNAMICS**

1. INTRODUCTION

The three body problem is one of the most important problems of celestial mechanics and has been analytically and numerically studied for centuries. In addition to that, three body interaction also plays an essential role for dynamics of binary and multiple stars. See Valtonen (2004) and Dvorak et al. (2004) and also their references.

On the other hand, because there are asteroid belt and Kuiper belt for the solar system, discs of dust for extrasolar planetary systems and also circumbinary rings for binary systems, these belt-like structures should influence the dynamical evolution of these systems. For instance, Jiang & Ip (2001) show that the origin of orbital elements of the planetary system of υ Andromedae might be influenced by the belt interaction initially. Moreover, Yeh & Jiang (2001) studied the orbital migration of scattered planets. They completely classify the parameter space and solutions and conclude that the eccentricity always increases if the planet, which moves on a circular orbit initially, is scattered to migrate outward. Thus the orbital circularization must be important for scattered planets if they are now moving on nearly circular orbits.

Therefore, Jiang & Yeh (2003) did some analysis on the solutions for dynamical systems of planet-belt interaction. In this paper, we further study the effect of belts for dynamical evolution of a binary system.

2. THE MODEL

We consider the motion of a test particle influenced by the gravitational force from the central binary and the circumbinary belt. The circumbinary belt also provides the frictional force for the test particle.

We assume that two masses of the central binary are m_1 and m_2 and choose the unit of mass to make $G(m_1+m_2)=1$. If we define that

$$\bar{\mu}=\frac{m_2}{m_1+m_2},$$

then the two masses are $\mu_1=Gm_1=1-\bar{\mu}$ and $\mu_2=Gm_2=\bar{\mu}$. The separation of the central binary is set to be unity and $\mu_1=\mu_2=0.5$ for all numerical results in this paper.

The equation of motion of this problem is (Murray & Dermott 1999)

$$\begin{cases} \frac{dx}{dt}=u \\ \frac{dy}{dt}=v \\ \frac{du}{dt}=2v-\frac{\partial U^*}{\partial x}-\frac{\partial V}{\partial x}+f_{\alpha x} \\ \frac{dv}{dt}=-2u-\frac{\partial U^*}{\partial y}-\frac{\partial V}{\partial y}+f_{\alpha y}, \end{cases} \tag{1}$$

where the potential U^* is

$$U^*=-\frac{1}{2}(x^2+y^2)-\frac{\mu_1}{r_1}-\frac{\mu_2}{r_2}, \tag{2}$$

$r_1=\sqrt{(x+\mu_2)^2+y^2}$ and $r_2=\sqrt{(x-\mu_1)^2+y^2}$.

V is the potential from the belt. The belt is a annulus with inner radius r_i and outer radius r_o,

[1] National Central Univ., Taiwan.
[2] National Hsinchu Teachers College, Taiwan.

where r_i and r_o are assumed to be constants. We arbitrarily set $r_i = 0.2$ and $r_o = 1.0$ for all results in this paper.

The density profile of the belt is $\rho(r) = c/r^p$, where $r = \sqrt{x^2+y^2}$, c is a constant completely determined by the total mass of the belt and p is a natural number. In this paper, we set $p = 2$ for all numerical results. Hence, for $p = 2$, the total mass of the belt is

$$M_b = \int_0^{2\pi}\int_{r_i}^{r_o} \rho(r')r'dr'd\phi = 2\pi c(\ln r_o - \ln r_i). \quad (3)$$

The gravitational force f_b from the belt is

$$f_b(r) = -\frac{\partial V}{\partial r} = -2\int_{r_i}^{r_o} \frac{\rho(r')r'}{r}\left[\frac{E}{r-r'} + \frac{F}{r+r'}\right] dr', \quad (4)$$

where $F(\xi)$ and $E(\xi)$ are elliptic integrals of the first and second kind. Hence,

$$\begin{cases} -\frac{\partial V}{\partial x} = f_b \frac{x}{r} \\ -\frac{\partial V}{\partial x} = f_b \frac{y}{r}, \end{cases} \quad (5)$$

where f_b is in Eq. (4).

The frictional force should be proportional to the surface density of the belt and the velocity of the particle. In the x direction, the frictional force is

$$f_{\alpha x} = -\alpha\rho(r)\frac{dx}{dt} \quad (6)$$

and in the y direction, the frictional force is

$$f_{\alpha y} = -\alpha\rho(r)\frac{dy}{dt}, \quad (7)$$

where α is the frictional parameter.

We substitute Eq. (2) and Eq. (4)-(7) into Eq.(1) and have the following system:

$$\begin{cases} \frac{dx}{dt} = u \\ \frac{dy}{dt} = v \\ \frac{du}{dt} = 2v + x - \frac{\mu_1(x+\mu_2)}{r_1^3} - \frac{\mu_2(x-\mu_1)}{r_2^3} - \frac{2x}{r^2} \\ \times \int_{r_i}^{r_o} \rho(r')r'\left[\frac{E}{r-r'} + \frac{F}{r+r'}\right] dr' - \alpha\rho(r)u \\ \frac{dv}{dt} = -2u + y - \frac{y\mu_1}{r_1^3} - \frac{y\mu_2}{r_2^3} - \frac{2y}{r^2} \\ \times \int_{r_i}^{r_o} \rho(r')r'\left[\frac{E}{r-r'} + \frac{F}{r+r'}\right] dr' - \alpha\rho(r)v. \end{cases} \quad (8)$$

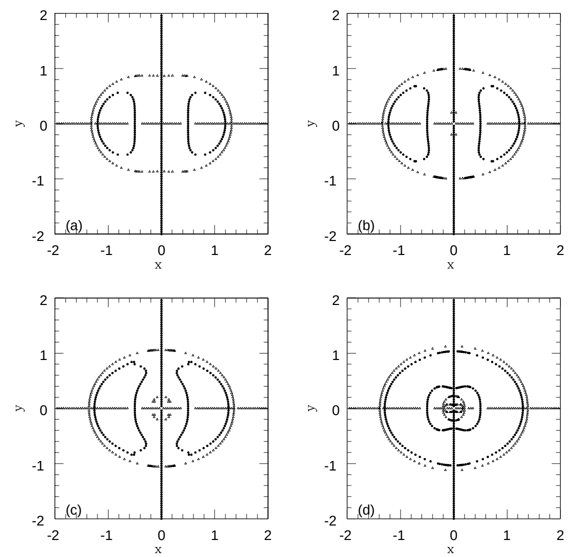

Fig. 1. The curves of $f(x,y) = 0$ and $g(x,y) = 0$ (see the text for details).

3. EQUILIBRIUM POINTS

The equilibrium points (x_e, y_e) of System (8) satisfy the following equations

$$f(x,y) \equiv x - \frac{\mu_1(x+\mu_2)}{r_1^3} - \frac{\mu_2(x-\mu_1)}{r_2^3}$$
$$-\frac{2x}{r^2}\int_{r_i}^{r_o} \rho(r')r'\left[\frac{E}{r-r'} + \frac{F}{r+r'}\right] dr' = 0, \quad (9)$$
$$g(x,y) \equiv y - \frac{y\mu_1}{r_1^3} - \frac{y\mu_2}{r_2^3} - \frac{2y}{r^2}$$
$$\int_{r_i}^{r_o} \rho(r')r'\left[\frac{E}{r-r'} + \frac{F}{r+r'}\right] dr' = 0. \quad (10)$$

In Fig. 1, we plot the curves of $f(x,y) = 0$ (circles) and $g(x,y) = 0$ (triangles) for different values of M_b. Equilibrium points (x_e, y_e) are intersection of these two. In Fig. 1(a), we set $M_b = 0$, so there is no influence from the belt. We find there are five usual Lagrangian points, L_1, L_2, L_3, L_4 and L_5 for this case. In Fig. 1(b), $M_b = 0.15$ and we still have the five usual Lagrangian points. In addition to that, there are two new equilibrium points in the upper half-plane and another two in the lower half-plane. In Fig. 1(c)-(d), we set $M_b = 0.3$ and $M_b = 0.5$ individually. We also find that in the upper half-plane, there are two new equilibrium points, F_a and F_b.

4. LYAPUNOV EXPONENT

Since we have discovered two new equilibrium points near L_4 (and another two near L_5), it would be interesting to investigate the orbital behavior around these new equilibrium points F_a and F_b. For a complicated system like ours, it is difficult to rigorously prove if the orbits are chaotic near F_a and F_b. Nevertheless, we use the calculation of the Lyapunov exponent for some orbits whose initial conditions are chosen to be close to F_a and F_b to understand how sensitively dependent on the initial conditions these orbits are. This is in fact one of the most important methods to study chaotic systems. We follow Wolf et al. (1985) to calculate the values of the Lyapunov exponent numerically. To check if our calculation is correct, we have reproduced the results of a given system in their paper.

In general, a larger value of the Lyapunov exponent means more sensitive dependence on the initial conditions. We choose the initial conditions of the orbits to be close to the equilibrium points F_a, F_b, L_4 and L_2 individually. Thus there are 4 different initial conditions for the orbital calculations. To understand the effects of belts with different masses, we did calculations for 4 different masses of the belt ($M_b = 0, 0.15, 0.3, 0.5$) for each chosen initial condition. Although there are no equilibrium points F_a, F_b when there is no belt ($M_b = 0$), and the locations of equilibrium point F_a, F_b, L_4 and L_2 would be slightly different for different masses of the belt, we still call the initial condition $(x, y, u, v) = (0.01, 0.0225, 0, 0)$ initial condition F_a, $(x, y, u, v) = (0.01, 0.06, 0, 0)$ initial condition F_b, $(x, y, u, v) = (0.01, 1, 0, 0)$ initial condition L_4 and $(x, y, u, v) = (1.35, 0, 0, 0)$ initial condition L_2.

Fig. 2(a)-(d) are the results of the Lyapunov exponent for initial condition F_a, F_b, L_4 and L_2 individually. There are 4 curves in each panel of Fig. 2, where the solid curve is the result of $M_b = 0$, the dotted curve is the result of $M_b = 0.15$, the dashed curve is the result of $M_b = 0.3$ and the long dashed curve is the result of $M_b = 0.5$. It is obvious that the values of Lyapunov exponent for initial condition F_a, F_b are much larger than the ones for initial condition L_4 and L_2. From panels (a) and (b), we can also see that the Lyapunov exponents for $M_b = 0.5$ and $M_b = 0.3$ are larger than the values for $M_b = 0.15$ and $M_b = 0$. Interestingly, for orbits with initial condition L_4, the values of the Lyapunov exponent are slightly larger for $M_b = 0$ as we can see in Fig. 2(c). In general, their values are small for both initial conditions L_4 and L_2. The values of the Lyapunov exponent for orbits with initial condition L_2 approach 0 when t tends to infinity. The orbits are obviously not chaotic for this case.

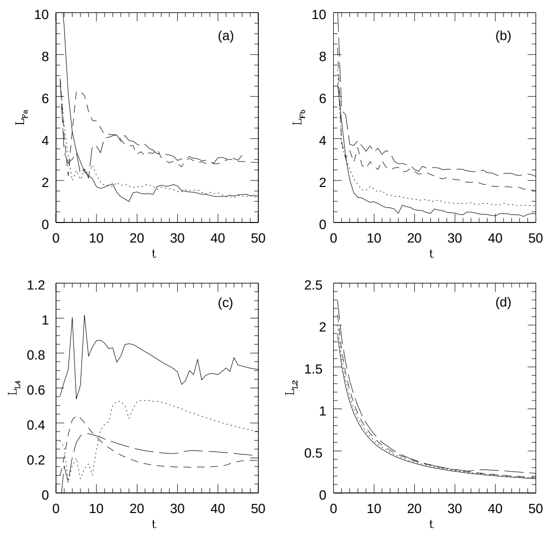

Fig. 2. Lyapunov exponent (see the text for details).

5. ORBITS

In this section, we will discuss all the orbits whose results of Lyapunov exponent have been shown and discussed in last section.

Figs. 3, 4, 5 and 6 are the orbits on $x - y$ plane for initial conditions F_a, F_b, L_4 and L_2. There are 4 panels for each figure. Panel (a) is the result when there is no belt, i.e. $M_b = 0$, panel (b) is the result when $M_b = 0.15$, panel (c) is for $M_b = 0.3$ and panel (d) is the result for $M_b = 0.5$.

If one looks at all these 4 figures of orbits at the same time, one can immediately understand that it seems the orbits with initial conditions F_a and F_b are much more chaotic than the orbits with initial conditions L_4 and L_2. This impression is completely consistent with the one we get from the values of the Lyapunov exponent.

To compare Fig. 3(a) with Fig. 4(a), we found that the orbits are similar for initial conditions F_a and F_b. However, from the comparison between Fig. 3(b) and Fig. 4(b), we found that the orbits are quite different for initial condition F_a and F_b. These two comparisons show that the existence of a belt does make the orbits become more sensitive to the initial conditions.

6. CONCLUDING REMARKS

We have provided the equations for a model which modifies the classical restricted three body problem by including the influence from a belt around the central binary. We found that, in addition to the usual Lagrangian points, there are two new equilibrium points, which we call F_a and F_b

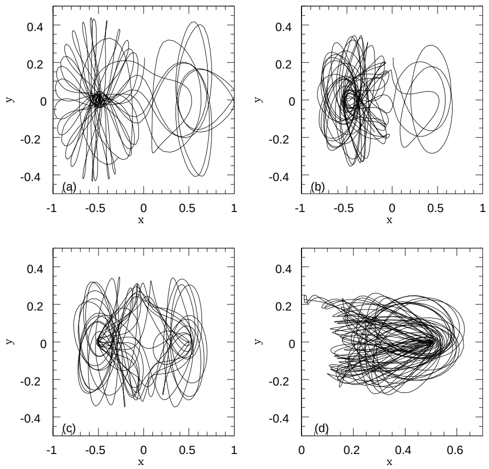

Fig. 3. The orbits with initial condition F_a.

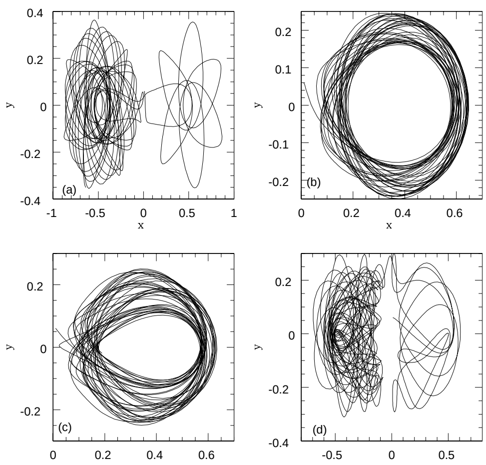

Fig. 4. The orbits with initial condition F_b.

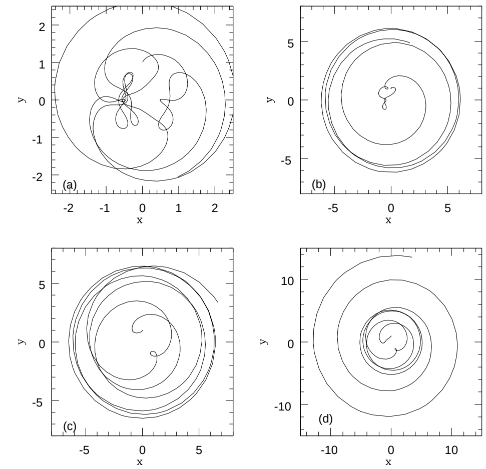

Fig. 5. The orbits with initial condition L_4.

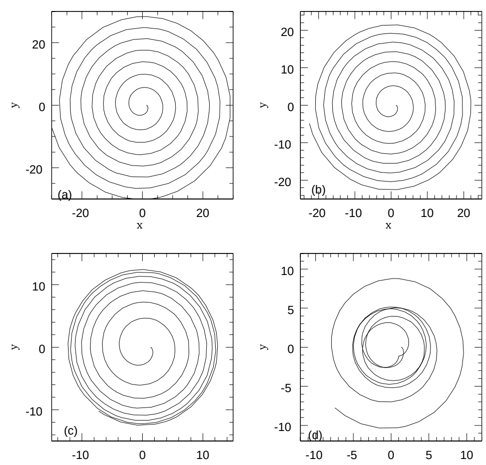

Fig. 6. The orbits with initial condition L_2.

around L_4 (similarly, there are another two new equilibrium points close to L_5).

To study the orbits around these new equilibrium points, we calculate the values of Lyapunov exponents for orbits with 4 different initial conditions, which are close to F_a and F_b, L_4 and L_2 individually. We found that the belt makes the system even more sensitive to the initial conditions for the orbits with initial conditions F_a and F_b but does not make too much difference for the orbits with initial conditions L_4 and L_2. Because the equilibrium points F_a and F_b happen to be near the inner part of the belt and Lagrangian points L_4 and L_2 happen to be around or out of the outer part of the belt in our system, it seems that the orbits near the inner part of the belt might be more unpredictable than the ones around the outer part.

REFERENCES

Dvorak, R., Pilat-Lohinger, E., Bois, E., Funk, B., Freistetter, F., & Kiseleva-Eggleton, L. 2004, this volume.

Jiang, I.-G., & Ip, W.-H. 2001, A&A, 367, 943.

Jiang, I.-G., & Yeh, L.-C. 2003, Int. J. Bifurcation and Chaos, 13, 534.

Murray, C. D. & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge University Press).

Valtonen, M. 2004, this volume.

Wolf, A., Swift, J. B., Swinney, H. L., & Vastano, J. A. 1985, Physica 16D, 285.

Yeh, L.-C., & Jiang, I.-G. 2001, ApJ, 561, 364.

Ing-Guey Jiang: Institute of Astronomy, National Central University, Taiwan (jiang@astro.ncu.edu.tw).
Li-Chin Yeh: Department of Mathematics, National Hsinchu Teachers College, Hsin-Chu, Taiwan (lcyeh@bsd.nhctc.edu.tw).