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A SIMPLE MODEL TO DETERMINE CHAOTIC MOTIONS AROUND ASTEROIDS

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RESUMEN

Es conocido que el movimiento tridimensional en las proximidades del asteroide (433) Eros es caótico para un amplio abanico de inclinaciones. La principal característica de este asteroide es su forma alargada, lo que hace que los procedimientos habituales de desarrollar el potencial en armónicos esféricos no tenga mucha utilidad. Por tal razón, consideramos un modelo simple que aproxime su forma alargada; en concreto, tomamos una varilla alargada y en rotación. De este modo sepuede obtener el potencial de forma cerrada, sin necesidad de desarrollos. Con este modelo tan simple y mediante el cálculo de familias de órbitas periódicas, podemos obtener una explicación para la mayoría de las características dinámicas previamente halladas para el movimiento orbital alrededor de Eros.

ABSTRACT

It is well known that the 3-D motion around the asteroid (433) Eros is chaotic for a wide set of inclinations. The main feature of this body is its elongated shape, which originates that the usual procedure of expanding the potential function in spherical harmonics is useless. This is the reason why we consider a simple model to describe its elongated shape, namely, a rotating finite straight segment. By so doing, it is possible to obtain the potential in closed form, with no need of further expansions. For this model, from the analysis of families of periodic orbits, we can find an explanation for most of dynamics already found for orbits around Eros.

Key Words: **CELESTIAL MECHANICS — MINOR PLANETS, ASTEROIDS**

1. INTRODUCTION

At present, there are space missions planned which targets are minor bodies in the Solar System, such as asteroids, comet nuclei or planet satellites, like Phobos or Deimos. Designing orbits around those bodies is a new challenge because of the uncertainty of some of its physical properties, like their mass, density and rotation, and also, because one of the main features of these bodies is their elongated shape. Taking into account the latter aspect, the classical way of expanding the gravity potential in Legendre polynomial series may diverge at some points (Balmino 1999). Consequently, a new question raised: how to efficiently represent the gravity field of such irregular bodies? To answer this question, several approaches have been proposed. Thus, Werner (1994) models the asteroid by a homogeneous polyhedron and derive formulas for the gravity field. This same model has been applied (Werner & Scheeres 1997) to asteroid 4769 Castalia. However, the polyhedron model, although very accurate, presents a lot of difficulties from the practical point of view, since there are many free parameters, there may be singularities at the corners and edges. This model makes difficult the computations of the derivatives to find the acting force. For some particular cases, other models, not so sophisticated, may give a good approximation for some bodies. In this way, Scheeres (1994) and Lara & Scheeres (2002) make the classical Legendre expansion and take only the first tesserals to analyze the dynamics around asteroid 433 Eros.

A different approach was made in Halamek (1988), although it already appears in Duboshin (1959). It consists of a finite straight segment with homogeneous distribution of mass. As its main advantage, it is possible to express its potential in closed form as a logarithm, depending on intrinsic quantities, namely, the length of the segment and the distances to the end points of the segment (Elipe et al. 1999; Riaguas et al. 1999). An extension of this model, considering two perpendicular straight segments, has been proposed very recently (Bartczak & Breiter 2003), and a comparison with respect to the expansion in spherical harmonics is made.

Asteroids and planetary satellites are old objects in the solar system and have reached the state of lowest energy for a given angular momentum, i.e., pure rotation about the principal axis of highest moment of inertia; any primeval nutation faded away because

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nutation induces time-varying internal stresses that dissipate mechanical energy through hysteresis cycles (see e.g. Prieto & Gómez 1994).

In this communication, we will briefly present some of aspects of the dynamics around a finite straight segment in rotation, and we will address the interested reader to the references.

The potential of a finite straight segment of length 2ℓ per unit of mass is given by

$$U = -\frac{GM}{2\ell}\log\left(\frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell}\right),$$

where G stands for the Gaussian constant and r_1 and r_2 are the distances from the point P to the end points of the segment. Thus, the potential is expressed in closed form in terms of intrinsic quantities, namely distances.

When the segment is uniformly rotating with angular velocity ω about an axis perpendicular to it and passing through its center of mass, and after an adequate choice of units, we can define the effective potential

$$W = -\frac{1}{2}(x^2 + y^2) - k\,\log\left(\frac{r_1 + r_2 + 1}{r_1 + r_2 - 1}\right), \qquad (1)$$

where $k = GM/(\omega^2(2\ell)^3)$; when $k < 1$ we have fast rotation of the segment. On the contrary, $k > 1$ means slow rotation with respect to the orbital mean motion.

In this rotating frame, the equations of motion are

$$\ddot{x} - 2\dot{y} = -W_x, \quad \ddot{y} + 2\dot{x} = -W_y, \quad \ddot{z} = -W_z, \quad (2)$$

that have the Jacobian integral $h = 2W + 2T$,with T the kinetic energy.

Stationary points give valuable information about the dynamics. It can be proved by analyzing the linearized equations (Riaguas 1999; Elipe et al. 1999) that Eqs. (2) have 4 equilibria, two on the x axis that are unstable for whatever value of the parameter k, and two on the y axis, unstable for $k \geq 4.548$, and linearly stable for $k < 4.548$. A further and more complex analysis, which requires the use of normal forms and some previous results for Hamiltonians with 2 degrees of freedom, proves that these points are indeed orbitally stable (Riaguas 1999). Besides, the analysis was extended for the resonant cases (Riaguas et al. 2001).

Equations (2) are highly nonlinear, and Poincaré sections show a transition order-chaos (Riaguas et al. 1999). This strong nonlinearity gives also some indications about their non-integrability, which indeed happens; in Arribas & Elipe (2001) it was proved that for any expansion of the potential, the system has no mero morphic integrals but the Hamiltonian itself.

But, we recall there are planned missions to orbit around those bodies and it should be desirable to split regions of initial conditions where the motion is regular from others regions where the motion is chaotic. This goal may be achieved by means of families of periodic orbits. Computing families of periodic orbits is a delicate task, and there are several ways to obtain them. In our case, we used the numerical continuation method given by Deprit & Henrard (1967) with the modifications given in Lara et al. (1995), based on a prediction-correction scheme. By so doing, from a starting periodic orbit, we can propagate the family by integrating jointly the equations of motion and the variational equations. In our case, as starting orbit, we take a circular orbit with very big radius (close to a Keplerian circle); then, by modifying slowly the Jacobian constant we generate the family that sprouts from the circular orbit.

The eigenvalues of the monodromy matrix (that is, the transition matrix evaluated after one period) arecomputed along the procedure, and provide the *stability index*, which is used to determine not only the stability of the orbits along the family, but the values of the bifurcations and the type of bifurcation that takes place, that is, if the family bifurcates on the plane or if it is a vertical bifurcation. For details about the stability index, see Broucke (1969); Hénon (1973); Lara & Peláez (2002). Since the problem is Hamiltonian, if λ_i is eigenvalue, $1/\lambda_i$ is also eigenvalue; besides, the system is autonomous, and one eigenvalue is the unit with multiplicity 2. Hence, two stability indices

$$k_1 = \lambda_1 + 1/\lambda_1, \qquad k_2 = \lambda_2 + 1/\lambda_2,$$

are defined in such a way that if $|k_i| < 2 \quad (i = 1, 2)$ we have linear stability for the orbit. Otherwise, we have instability.

It is known that bifurcations from stability to instability are the threshold to chaos. Thus, we proceeded in Elipe & Lara (2003) to determine these bifurcation lines. First, we computed periodic orbits in a rotating frame attached to the segment and determined their stability, which helps in separating regions of stable trajectories from escape or collision ones. In this way, we extended the work (Elipe et al. 1999) by computing the vertical stability (k_2) of planar periodic orbits. We found that three-dimensional periodic orbits are members of bifurcated families of the planar equatorial family of periodic orbits — we call "equator" the plane perpendicular to the rotation axis. We also noticed that these bifurcations appear when there are commensurabilities between the rotation rate of the segment and the mean motion of the orbiter in an inertial frame, which suggested that these resonances would play an important role on the bifurcations and on the stability. Thus, we decided to tackle the problem of the resonances among these angular velocities.

Let us assume that there is commensurability between the two frequencies involved in the problem, namely, the orbital period in the inertial frame and the rotating period of the segment. Let us assume that the orbiter gives N revolutions in the inertial frame around the segment while this rotates D times around its rotation axis. Then, for retrograde orbits there results that $-n/\omega = N/D$, and if we denote by $P_{\rm R} = 2\pi/|\tilde{n}|$ the period of the orbit in the synodic frame, there results that

$$P_{\rm R} = \frac{2\pi}{1+N/D},$$

The orbit will close in the inertial frame after a time $\Delta t = 2\pi D = (D+N)P_{\rm R}$ producing the resonance. Therefore, the D:N-resonant orbit is obtained by locating the orbit with period $P_{\rm R}$ on the retrograde family. The corresponding initial conditions are used then with multiple period $P = (D+N)P_{\rm R}$, as starter of thepredictor-corrector algorithm. By propagating the family with respect to the Jacobian constant h, we obtain two types of bifurcations, namely vertical bifurcation for increasing values of h and horizontal bifurcation for smaller values of h. Once the vertical bifurcation has been found, slight variations in the initial conditions will result in a three-dimensional periodic orbit of the vertically bifurcated family.

The computation of different resonances occurring at different distances from the orbiter to the origin provides an effective way to get very close to the critical points with multiple period where the corresponding branches of families of three-dimensional periodic orbits bifurcate. Further, the continuation of the vertically bifurcated families of periodic orbits and the computation of the stability indices enable the determination of a set of points, defining a line in phase space where the stability of the periodic orbits changes. A detailed explanation can be found in Elipe & Lara (2003).

The continuation of the families of three-dimensional periodic orbits that vertically bifurcate in the vicinity of different resonances of planar retrograde motion (Fig. 1), as well as the computation

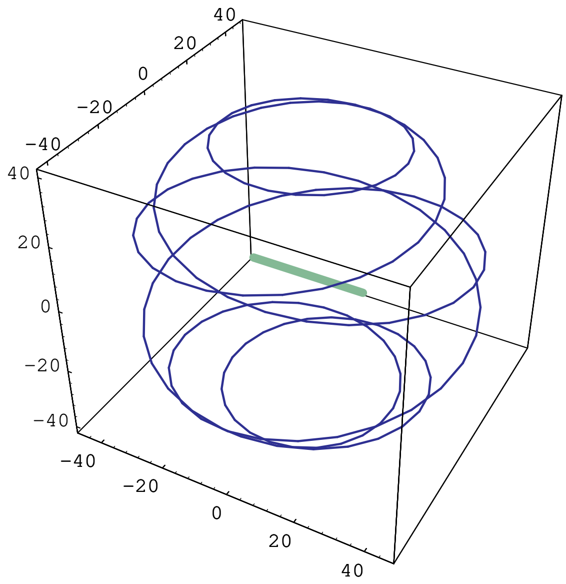

Fig. 1. Sample orbit of the 1:5-resonance family in the rotating frame.

of the stability indices enable the determination of a set of inclination values where almost circular orbits change from stability to instability. We can plot these values versus the distance to the origin in a 2D graphic (or equivalently versus the resonance value) and have a set of points that, if dense enough, define a line in the phase space where the stability of the periodic orbits changes.

By so doing, we obtain the Figure 2 that presents a set of points corresponding to the critical almost circular orbits that are periodic in the cycles (nodal periods/nodal days) 1:1, 4:5, 3:4, 2:3, 3:5, 4:7, 1:2, 3:7, 3:8, 1:3, 1:4, and 1:5. Ordinates present the inclination I (the average along one period), while abscissas are the inverse of the repetition cycle (1:1, 5:4, . . . , 5:1), that is directly related to the distance of the orbit to the origin. In this figure, we find stability for most retrograde orbits, while instability appears for direct motion close to the origin. Joining these dots, we can define three stability regions:

1. In the first region, we find stability for almost circular orbits.

2. In the second region, we find instability for almost circular orbits. The motion could be easily controlled since the stability indices are not very high.

3. In the third region, the instability is very high, leading very soon to escape or collision orbits.

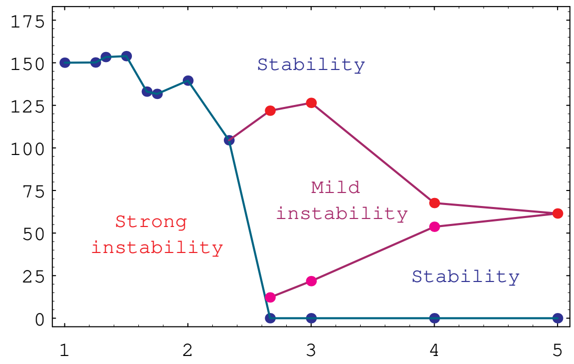

Fig. 2. Stability indices k_1, k_2 of the family bifurcated at the 1:1-resonance as a function of the inclination I and the inverse of the repetition cycle.

With this simple model, our results are in good agreement with the ones obtained with other techniques, such as the ones given in Scheeres et al. (1999).

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