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# ARTIFICIAL SATELLITE THEORY: CONTRIBUTION OF THE TESSERAL HARMONIC COEFFICIENTS

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RESUMEN

En el marco de la teoría del satélite artificial, se presenta aquí el estudio del caso de un satélite artificial moviéndose alrededor de un planeta similar a la Tierra. La fuerza de perturbación considerada es la debida a la inhomogeneidad del potencial gravitacional del planeta. El análisis que se presenta es puramente analítico y considera el truncamiento de las ecuaciones Hamiltonianas a partir del segundo orden, es decir, la influencia de los armónicos zonales y teserales hasta segundo orden. El sistema se formula como un Hamiltoniano con tres grados de libertad. Por medio de tres transformaciones de Lie sucesivas y sin hacer uso de expansiones de Taylor para la excentricidad ni de series de Fourier para las anomalías, el sistema se normaliza hasta orden cinco en un pequeño parámetro. El Hamiltoniano resultante define un sistema de un grado de libertad, que puede ser expresado en el espacio de fases en términos de una colección de variables que tienen en cuenta todas las simetrías del problema. El análisis de este sistema da como resultado la obtención de cuatro regiones de equilibrio relativo y dos tipos de bifurcaciones. La comparación de este problema reducido con el sistema original nos muestra la existencia de varias familias de toroides invariantes, trayectorias periódicas y cuasiperiódicas con el mismo tipo de bifurcaciones.

ABSTRACT

In the context of artificial satellite theory, we study the case of an artificial satellite around an Earth-like planet. The perturbation force taken into account is produced by the gravity attraction of the planet, and in particular, it is caused by its inhomogenous potential. Our analysis is purely analytical and we begin by truncating the Hamiltonian equations at second order, that is, considering the influence of the zonal and the tesseral harmonics to order two. The system is formulated as a Hamiltonian with three degrees of freedom. Then, by means of three successive Lie transforms, we normalise the system to order five in a small parameter. We do not use Taylor expansions of the eccentricity nor Fourier series in the anomalies. After these transformations are performed, the truncated and reduced Hamiltonian defines a system of one degree of freedom which can be written in terms of a collection of variables in a phase space which takes into account all of the symmetries of the problem. Next, an analysis of the system is performed, obtaining up to four relative equilibria and two types of bifurcations. The connection with the original system establishes the existence of various families of invariant tori, quasiperiodic and periodic trajectories with the same type of bifurcations as the reduced problem.

*Key Words:* **CELESTIAL MECHANICS — PLANETS AND SATELLITES: GENERAL**



## 1. SCOPE

The gravity field of a planet is the biggest perturbation that affects a satellite. In general, analytical theories are employed to provide fast and accurate calculation of ephemeris, although for a satellite orbiting at low altitude these theories are normally used to study the time variation of some tesseral coefficients of the gravity field. This paper deals with the influence of the tesserals in the motion of a satellite orbiting an Earth-like planet at low altitude.

The tesseral problem has been studied by various authors since Kaula (1966). In most of the works, the entire perturbing potential, that is, including the zonals and tesserals coefficients, is placed at first order of perturbation, applying thereafter canonical transformations with the aim of eliminating the mean anomaly at first order of perturbation (Métris et al. 1993). These approaches are usually done in a phase space free of resonances.

Our treatment is based on a scaling of the Hamiltonian, so that we take into account the relative values of the coefficients involved in the process. In particular, we assume that the oblateness coefficient, i.e., the zonal term $C_{2,0}$, is much bigger than the rest of terms as is the case for the Earth. Thus, we are able to remove both short and long period terms from the equations of motion applying Lie transforms (Deprit 1969). We choose the small parameter as the quotient between the angular velocity of the

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planet and the mean motion of the satellite and arrange the initial Hamiltonian in a convenient way to perform the transformations.

Once the Hamiltonian is simplified, we use reduction techniques to express it in an adequate set of coordinates and its appropriate phase space where the flow may be discussed. In this way, we enlarge the work of Cushman (1983, 1984) and Coffey et al. (1986) for the zonal problem, finding the relative equilibria of the reduced Hamiltonian, and analysing its (non-linear) stability and bifurcations. In this article, we present the main features of our theory while the pertinent details will appear in the near future (Palacián 2005).

## 2. HAMILTONIAN OF THE PROBLEM

We choose two sets of variables well suited to perform the normalisation of our original Hamiltonian, the so-called *polar-nodal* and *Delaunay* variables. For an explanation of both sets of canonical variables, see Deprit (1981). We start by fixing an inertial frame, say $0\,x\,y\,z$, centered at the centre of mass of the planet. Polar-nodal variables belong to the set $(r, \vartheta, \nu, R, \Theta, N)$, where $\Theta$ is the modulus of the angular momentum vector, $\boldsymbol{G}$, and its conjugate angle is the argument of the latitude $\vartheta \in [0, 2\pi)$. The co-ordinate $r$ is the distance from the centre of the planet to the satellite and its conjugate momentum $R$ denotes the radial velocity. The argument of the node $\nu$ is the angle conjugate to the action $N$, which represents the projection of $\boldsymbol{G}$ onto the $z$–axis. The inclination of the instantaneous orbital plane with respect to the $x\,y$–plane (the so-called equatorial plane) is given by the angle $0 < I < \pi$ such that $N = \Theta\,\cos(I)$. We define $c = \cos(I)$ and $s = \sin(I)$.

On the other hand, Delaunay variables $(\ell, g, h, L, G, H)$ represent a set of action-angle variables, see (Deprit 1981,1982) for details. The action $L$ is related with the semimajor axis of the orbit by the identity $L^2 = \mu\,a$ where the gravitational constant is $\mu$. Thence, if $\mathcal{H}_0$ stands for the Hamiltonian of the two-body problem, $\mathcal{H}_0 = -\mu^2/(2\,L^2)$. The action $G$ is equal to $\Theta$, whereas $H \equiv N$. The angle $\ell$ stands for the mean anomaly. The angle $g$ is the argument of pericentre and $h \equiv \nu$. The eccentricity of the trajectory is designated by $e$, and in terms of Delaunay actions, it is expressed as $e = (1 - G^2/L^2)^{1/2}$.

If the planet is assumed to rotate with a uniform angular speed $\omega$ one can choose a three-dimensional reference frame attached to the planet in a way that its $z'$ component corresponds to the axis of rotation. Next, the Hamiltonian of the problem can be written as the sum $\mathcal{H} = \mathcal{T} - \omega\,N + \mathcal{V}$, where $\mathcal{T}$ and $\mathcal{V}$ represent, respectively, the kinetic and the potential energies. In particular $\mathcal{V}$ reads as

$$\mathcal{V} = -\frac{\mu}{r}\left[1 + \left(\frac{\alpha}{r}\right)^2 \mathcal{V}_2 + \left(\frac{\alpha}{r}\right)^3 \mathcal{V}_3 + \left(\frac{\alpha}{r}\right)^4 \mathcal{V}_4 + \ldots\right],$$

where $\alpha$ stands for the mean equator of the planet. If we drop the coefficients of order higher than two, the remaining one reduces to

$$\begin{aligned}
\mathcal{V}_2 = \tfrac{1}{2}(\tfrac{3}{2}s^2 - 1) &\quad C_{20} \\
- \tfrac{3}{4}s^2 &\quad C_{20}\cos 2\vartheta \\
- \tfrac{3}{4}(c-1)s &\quad (S_{21}\cos(\nu - 2\vartheta) - C_{21}\sin(\nu - 2\vartheta)) \\
+ \tfrac{3}{2}cs &\quad (S_{21}\cos\nu - C_{21}\sin\nu) \\
- \tfrac{3}{4}(c+1)s &\quad (S_{21}\cos(\nu + 2\vartheta) - C_{21}\sin(\nu + 2\vartheta)) \\
+ \tfrac{3}{4}(c-1)^2 &\quad (C_{22}\cos(2\nu - 2\vartheta) + S_{22}\sin(2\nu - 2\vartheta)) \\
+ \tfrac{3}{2}s^2 &\quad (C_{22}\cos 2\nu + S_{22}\sin 2\nu) \\
+ \tfrac{3}{4}(c+1)^2 &\quad (C_{22}\cos(2\nu + 2\vartheta) + S_{22}\sin(2\nu + 2\vartheta))\,.
\end{aligned}$$

The goal of the following paragraphs is to normalise and reduce the (autonomous) three-degree-of-freedom system defined by $\mathcal{H}$ into a new system of one degree of freedom.

## 3. NORMALISATIONS AND REDUCTIONS

We need to pass from $\mathcal{H}$ to a new Hamiltonian $\mathcal{K}$ using Lie transforms. We make

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \frac{1}{2}\mathcal{H}_2 + \frac{1}{6}\mathcal{H}_3$$

where $\mathcal{H}_0$ corresponds to the Hamiltonian of the two-body problem, $\mathcal{H}_1 = -\omega\,N$, $\mathcal{H}_2$ contains the terms factored by $C_{20}$ and $\mathcal{H}_3$ have the terms related to the tesseral coefficients. Higher-order terms are taken equal to zero. By doing so, the small parameter of the problem is considered to be the size of $\omega/n$.

Next, we identify $\mathcal{K}_0 \equiv \mathcal{H}_0$ and apply three Lie transforms to obtain:

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 + \frac{1}{2}\mathcal{K}_2 + \frac{1}{6}\mathcal{K}_3 + \frac{1}{24}\mathcal{K}_4 + \frac{1}{120}\mathcal{K}_5.$$

The mean anomaly is removed through two successive Lie transforms: (i) following Deprit (1981), we apply the *elimination of the parallax* to alleviate the number of terms in the resulting Hamiltonian; then, (ii) next $\ell$ is eliminated through a *Delaunay normalisation* (Deprit 1982). Both transformations are in closed form. Moreover, the removal of $\ell$, avoiding Taylor and Fourier expansions has been possible thanks to the introduction in the generating function of the Delaunay normalisation of the polylogarithmic function, $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n$, and some other combinations involving logarithmic and polylogarithmic terms (Osácar & Palacián 1994; Palacián 2002). Finally, the argument of the node is eliminated via a standard average. After truncating higher-order terms, the resulting Hamiltonian is independent of $\ell$ and $h$ and so $L$ and $H$ are integrals of motion for it. Thus, $\mathcal{K}$ defines a system of one degree of freedom. The reason for pushing the calculations to fifth order is that we need $\mathcal{K}_5$ to capture the influence of $C_{2,1}$, $C_{2,2}$, $S_{2,1}$ and $S_{2,2}$.

Now, we simplify $\mathcal{K}$ further. In fact, the discrete symmetries of $\mathcal{H}$ are inherited by $\mathcal{K}$:

$$\mathcal{R}_1 : (x, y, z, P_x, P_y, P_z) \to (x, -y, -z, -P_x, P_y, P_z),$$
$$\mathcal{R}_2 : (x, y, z, P_x, P_y, P_z) \to (x, -y, z, -P_x, P_y, -P_z).$$

Thus, one can define a couple of variables, $\sigma_1$ and $\sigma_2$, to reflect this symmetry. The relationship among the Delaunay and the new variables is as follows: $G = \sigma_2$. On the other hand, $\cos g$ is given by:

$$\pm\sqrt{\frac{L^2 H^2 - 4\,\sigma_1\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(\sigma_1 + 2\,\sigma_2^2)}{5\,L^2 H^2 - 4\,(L^2+H^2)\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(L^2+H^2-2\,\sigma_2^2)}}$$

whereas $\sin g$ yields:

$$\pm\sqrt{\frac{4\,L^2 H^2 - 4\,(L^2+H^2-\sigma_1)\,\sigma_2^2 - 2\,L\,|H|\,(L^2+H^2-\sigma_1-4\,\sigma_2^2)}{5\,L^2 H^2 - 4\,(L^2+H^2)\,\sigma_2^2 + 4\,\sigma_2^4 - 2\,L\,|H|\,(L^2+H^2-2\,\sigma_2^2)}}.$$

In this way, $\mathcal{K}$ is expressed in terms of $\sigma_1$ and $\sigma_2$, resulting in $\bar{\mathcal{K}}$. Then, this (averaged and fully-reduced) Hamiltonian is defined on the so-called fully-reduced phase spaces. For $|H| > 0$, the space $\mathcal{U}_{L,H}$ is given by:

$$\mathcal{U}_{L,H} = \Big\{(\sigma_1, \sigma_2) \in \boldsymbol{R}^2 \mid |H| \le \sigma_2 \le L, \ \frac{(\sigma_2^2 - L\,|H|)^2}{\sigma_2^2} \le \sigma_1 \le (L - |H|)^2\Big\},$$

whereas for $H = 0$, the space $\mathcal{U}_{L,0}$ is given through:

$$\mathcal{U}_{L,0} = \{(\sigma_1, \sigma_2) \in \boldsymbol{R}^2 \mid \sigma_2^2 \le \sigma_1 \le L^2,\ 0 \le \sigma_2 \le L\}.$$

## 4. ANALYSIS OF THE REDUCED SYSTEM

The flow defined by $\bar{\mathcal{K}}$ is analysed in $\mathcal{U}_{L,H}$ and $\mathcal{U}_{L,0}$. First, we concluded that circular and equatorial type of orbits are always relative equilibria. The rest of equilibria depend on five bifurcation curves. The lines $\Gamma_1, \Gamma_2, \Gamma_3$ and $\Gamma_4$ are pitchfork bifurcations, whereas $\Gamma_5$ is a special bifurcation due to the change of phase space (see Fig.1).

The main features of the analysis of $\bar{\mathcal{K}}$ are as follows: (i) An equilibrium on $\mathcal{U}_{L,H}$, whose linearisation has no null eigenvalue is in correspondence with

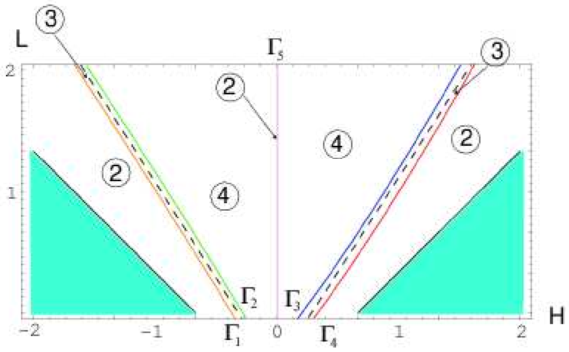


Fig. 1. Bifurcation diagram with the number of relative equilibria encircled in each region. Dot lines correspond to the critical inclination value for the zonal problem.

one or two families of two-dimensional invariant tori in $\boldsymbol{R}^6$ (parameterised by $L$ and $H$). These families share the same type of stability. (ii) Explicit formulae of the approximations of the two-dimensional invariant tori, quasiperiodic and periodic orbits are computed using the direct change of the Lie transforms. (iii) The analysis is valid for satellites orbiting the planet at low altitude ($\omega \ll n$). (iv) The inclusion of the tesseral coefficients modifies slightly the equations of the curves $\Gamma_1, \ldots, \Gamma_4$ of the zonal problem.

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