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ACCRETION MODEL OF A ROTATING GAS SPHERE ONTO A SCHWARZSCHILD BLACK HOLE

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RESUMEN

Construimos un modelo simple de acreción de una esfera de gas sin presión hacia un agujero negro de Schwarzschild. Lejos del agujero, el flujo rota como un cuerpo rígido. Mostramos como construir soluciones analíticas en términos de las funciones elípticas de Jacobi. Esta construcción representa la generalización relativista del modelo Newtoniano de acreción primeramente propuesto por Ulrich (1976). De la misma manera que, como ocurre en el caso Newtoniano, el flujo naturalmente predice la existencia de un disco de acreción ecuatorial rotando alrededor del agujero negro. Sin embargo, el radio del disco se incrementa de manera monotónica sin límite a medida que el flujo se acerca al máximo momento angular permitido por el modelo.

ABSTRACT

We construct a simple accretion model of a rotating pressureless gas sphere onto a Schwarzschild black hole. Far away from the hole, the flow is assumed to rotate as a rigid body. We show how to build analytic solutions in terms of Jacobi elliptic functions. This construction represents a general relativistic generalization of the Newtonian accretion model first proposed by Ulrich (1976). In exactly the same form as it occurs for the Newtonian case, the flow naturally predicts the existence of an equatorial rotating accretion disk about the hole. However, the radius of the disk increases monotonically without limit as the flow reaches the angular momentum corresponding to the maximum limit allowed by the model.

Key Words: accretion, accretion disks — hydrodynamics — relativity

1. INTRODUCTION

Steady spherically symmetric accretion onto a central gravitational potential (e.g. a star) was first investigated by Bondi (1952). A general relativistic generalization of this work was made by Michel (1972). However, realistic models of spherical accretion must consider that gas clouds where objects are embedded have a certain degree of rotation. The rotation of the gas cloud predicts the formation of an equatorial accretion disk for which gas particles rotate about the central object. The first steady accretion model, in which a rotating gas sphere with infinite extent is accreted to a central object was first investigated by Ulrich (1976). He made a ballistic analysis, which is approximately true if the initial specific angular momentum of an infalling particle is small and if heating by radiation and viscosity effects are negligible.

A first order general relativistic approximation of a rotating gas sphere was made by Beloborodov & Illarionov (2001). In their model, they use approximate solutions to the integration of the geodesic equation (cf. equation (1)) and the initial conditions of it are such that the specific angular momentum for a single particle $h \leq 2r_{\rm g}$, where $r_{\rm g}$ is the Schwarzschild radius. Such a model is not an appropriate generalization, since a correct one must satisfy that $h \geq 2r_{\rm g}$. A pseudo–Newtonian Paczynsky & Wiita (1980) numerical approximation of the ultra–relativistic $h = 2r_{\rm g}$ case was discussed by Lee & Ramirez-Ruiz (2006). This relativistic numerical approximation differs in a significant way from the complete general relativistic solution.

2. AN EXACT SOLUTION FOR THE ACCRETION PROBLEM IN GENERAL RELATIVITY

The geodesic equation for material particles in a Schwarzschild spacetime is given by

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\phi}\right)^2 = \alpha v^3 - v^2 + 2v + \epsilon, \qquad (1)$$

where

$$\alpha := 2\left(\frac{M}{h}\right)^2, \qquad \epsilon := \frac{2E_{\rm tot}h^2}{M^2}. \qquad (2)$$

This equation governs the geometry of the orbits described in the invariant plane $\theta = \pi/2$ due to

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the fact that the geometry of the geodesics is determined by the roots of the cubic equation $f(v) = \alpha v^3 - v^2 + 2v + \epsilon$.
A direct integration of equation (1) yields (cf. Huerta & Mendoza 2007)

$$r = \frac{p}{v_2\left(1 - \mathrm{cn}^2\varphi\beta\right)}, \tag{3}$$

where

$$\beta := \frac{(\alpha v_3)^{1/2}}{2} = \left(\frac{1 + (1 - 8\alpha)^{1/2}}{8}\right)^{1/2},$$
$$\text{and} \quad p = r_* \sin^2\theta_0,$$

where p is the *latus rectum* of the generalized conic. In the Newtonian limit, the length r_* converges to the radius of the Newtonian disk $r_{\mathrm{dN}} = h_\infty^2/M$.

In order to obtain an equation of motion in terms of the polar coordinate θ and the initial polar angle θ_0 made by a particle when it starts falling onto the black hole, we note the fact that in order to recover the geometry of the spherical 3D space as $\alpha \to 0$ it should be fulfilled that

$$\mathrm{cn}^2\varphi\beta = \frac{\mathrm{cn}^2\theta_0\beta + \mathrm{cn}^2\theta\beta - 1}{2\mathrm{cn}^2\theta_0\beta - 1}. \tag{4}$$

Therefore the orbit equation is given by

$$r = \frac{r_* \sin^2\theta_0\left(2\,\mathrm{cn}^2\theta_0\beta - 1\right)}{v_2\left(\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta\right)}. \tag{5}$$

The equations for the streamlines $r(\theta)$, the velocity field $v_r,\ v_\theta, v_\varphi$ and the proper particle number density n in terms of dimensionless variables (cf. Huerta & Mendoza (2007)) are given by

$$r = \frac{\sin^2\theta_0\left(2\mathrm{cn}^2\theta_0\beta - 1\right)}{v_2\left(\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta\right)}, \tag{6}$$

$$v_r = -2r^{-1/2}\beta\ \frac{\mathrm{cn}\beta\theta\ \mathrm{sn}\beta\theta\ \mathrm{dn}\beta\theta}{\sin\theta}\ f_1^{1/2}\left(\theta,\,\theta_0,\,v_2,\,\beta\right), \tag{7}$$

$$v_\theta = r^{-1/2}\frac{\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta}{\sin\theta}\ f_1^{1/2}\left(\theta, \theta_0, v_2, \beta\right), \tag{8}$$

$$v_\varphi = r^{-1/2}\frac{\sin\theta_0}{\sin\theta}\left(\frac{v_2\ \left(\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta\right)}{2\mathrm{cn}^2\theta_0\beta - 1}\right)^{1/2}, \tag{9}$$

$$n = \frac{r^{-3/2}\sin\theta_0}{2f_1^{1/2}\left(\theta,\,\theta_0,\,v_2,\,\beta\right)\ f_2\left(\theta,\,\theta_0,\,v_2,\,\beta\right)}, \tag{10}$$

where the functions $f_1\left(\theta, \theta_0, v_2, \beta\right)$ and $f_2\left(\theta, \theta_0, v_2, \beta\right)$ are defined by the following relations:

$$\begin{aligned}
f_1 &:= T1/T2,\\
T1 &:= 2\,\sin^2\theta\left(2\mathrm{cn}^2\theta_0\beta - 1\right)\\
&\qquad - v_2\,\sin^2\theta_0\left(\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta\right)\\
T2 &:= \left(2\,\mathrm{cn}^2\theta_0\beta - 1\right)\left\{\left(\mathrm{cn}^2\theta_0\beta - \mathrm{cn}^2\theta\beta\right)^2\right.\\
&\qquad \left. + \left(2\,\beta\mathrm{cn}\beta\theta\,\mathrm{sn}\beta\theta\,\mathrm{dn}\beta\theta\right)^2\right\},\\
f_2 &:= \ \beta\mathrm{cn}\beta\theta_0\ \mathrm{sn}\beta\theta_0\ \mathrm{dn}\beta\theta_0 + \left\{\sin\theta_0\ \cos\theta_0\right.\\
&\quad \times\left(2\mathrm{cn}^2\theta_0\beta - 1\right) - 2\beta\mathrm{cn}\beta\theta_0\ \mathrm{sn}\beta\theta_0\ \mathrm{dn}\beta\theta_0\\
&\quad \left.\times\sin^2\theta_0\right\}/v_2 r.
\end{aligned}$$

Equations (6)–(10) are the solutions to the problem of a rotating gas sphere onto a Schwarzschild black hole, i.e. they represent a relativistic generalization of the accretion model first proposed by Ulrich (1976).

This model converges to Ulrich accretion model when $\alpha \to 0$ (Huerta & Mendoza 2007). On the other hand, if we consider the particular case for which the angular momentum is null, then (6)–(10) describe a radial accretion model onto a Schwarzschild black hole. These equations correspond to the model first constructed by Michel (1972) for a pressureless fluid.

In addition, when the parameter α reaches its maximum value $\alpha = 1/8$, cf. equation (2), then this model does not formally represent a relativistic Ulrich solution, since the orbit followed by a particular fluid particle have a hyperbolic Newtonian counterpart. These solutions are describe in detail in Huerta & Mendoza (2007) and correspond to the exact relativistic solutions to the problem discussed by Lee & Ramirez-Ruiz (2006) and solved numerically using a Paczynsky & Wiita (1980) pseudo–Newtonian potential.

3. DISCUSSION

The work presented here represents a general relativistic approach to the ballistic Newtonian accretion flow first proposed by Ulrich (1976). The main features (see for example Figure 2) of the accretion flow are still valid with the important consequence that the radius of the equatorial accretion disk grows from its Newtonian value for the Ulrich case up to infinity in the extreme ultra–relativistic situation, for which the angular momentum is twice the Schwarzschild radius. As a consequence, the particle number density diverges on the border of the disk only for the Newtonian case described by Ulrich. As Figure 1 shows, the divergence of the par-

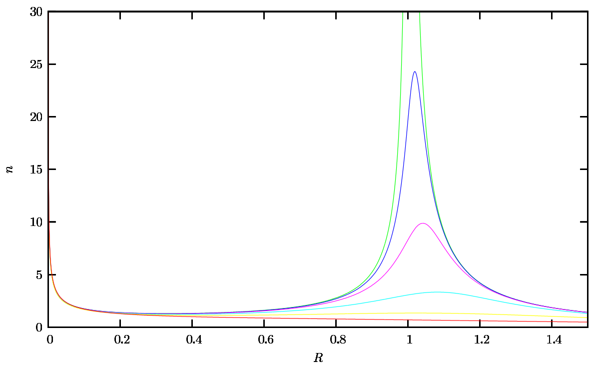

Fig. 1. A characteristic plot of particle number density n measured in units of n_0, as a function of the radial distance R measured in units of r_* evaluated in the equator, i.e. for which the polar angle $\theta = \pi/2$. This particular plot correspond to $\alpha = 10^{-1}$.

ticle number density at the border of the disk disappears as soon as α moves away from a null value. Furthermore, it does so in such a way that the density of the disk varies very smoothly throughout the disk as $\alpha \to 1/8$.

This is due to the fact that, when the radius of the disk grows, the particle number density on it rearranges in such a way that it smoothly softens as the $\alpha \to 1/8$. Figure 2 shows density isocontours for different values of the parameter α.

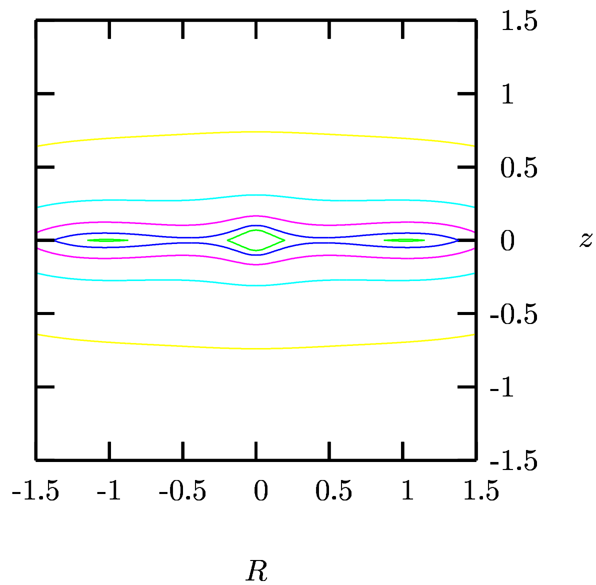

Fig. 2. Particle number density isocontours for $\alpha = 10^{-5}$, 0.05, 0.12 are shown in each diagram. Lengths in the plot are measured in units of the radius r_* and the density isocontours correspond to values of $n/n_0 =$ 0.1, 0.6, 1.1, 1.6, 2.1, 2.6.

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