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NEUTRON STARS AND THEIR MAGNETIC FIELDS

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RESUMEN

Las estrellas de neutrones poseen los campos magnéticos más intensos que conocemos. Aquí se pretende dar una discusión pedagógica de parte de la física relacionada. Las estrellas de neutrones existen gracias al principio de exclusión de Pauli, en dos sentidos: (1) Éste hace difícil comprimir muchas partículas en un espacio pequeño, permitiendo un estado de equilibrio mecánico en presencia de una fuerza de gravedad muy intensa. (2) La ocupación de estados cuánticos de baja energía por parte de protones y electrones impide el decaimiento beta de neutrones poco energéticos. Un corolario de esto es que necesariamente hay partículas cargadas en la estrella, permitiendo el flujo de corrientes eléctricas. Como las partículas son degeneradas, colisionan muy poco, haciendo posible la existencia prolongada de campos magnéticos intensos. Estos se revelan en pulsares y son la más probable fuente de energía para la alta luminosidad en rayos X y gamma en "magnetares". Discuto brevemente el posible origen de estos campos magnéticos, así como algunas consideraciones físicas que restringen a sus configuraciones de equilibrio.

ABSTRACT

Neutron stars have the strongest magnetic fields known anywhere in the Universe. In this review, I intend to give a pedagogical discussion of some of the related physics. Neutron stars exist because of Pauli's exclusion principle, in two senses: (1) It makes it difficult to squeeze particles too close together, in this way allowing a mechanical equilibrium state in the presence of extremely strong gravity. (2) The occupation of low-energy proton and electron states makes it impossible for low-energy neutrons to beta decay. A corollary of the second statement is that charged particles are necessarily present inside a neutron star, allowing currents to flow. Since these particles are degenerate, they collide very little, and therefore make it possible for the star to support strong, organized magnetic fields over long times. These show themselves in pulsars and are the most likely energy source for the high X-ray and gamma-ray luminosity "magnetars". I briefly discuss the possible origin of this field and some physical constraints on its equilibrium configurations.

Key Words: dense matter — magnetic fields — MHD — stars: magnetic fields — stars: neutron — white dwarfs

1. INTRODUCTION

This text aims at giving a pedagogical introduction to the physics of neutron stars and their magnetic fields, starting from undergraduate physics (Quantum Mechanics and Electromagnetism) and leading up to some current research questions such as the origin and equilibrium configuration of the magnetic field and clarifying some misconceptions appearing in the recent literature. All equations used here can be derived "on the back of an envelope", based only on undergraduate physics, and students are strongly encouraged to do so.

In the philosophy of keeping the treatment simple and aiming at a good understanding of basic questions rather than introducing all the complications that might conceivably arise, I consider an extremely conservative model of neutron stars that includes neutrons, protons, and electrons, all treated as degenerate and mostly non-interacting fermions. This neglects many extremely interesting (but uncertain) issues such as neutron superfluidity, proton superconductivity, or quark deconfinement and condensation. For more comprehensive treatments of neutron star physics, also addressing many of the "exotic" issues, I suggest the classic book of Shapiro & Teukolsky (1983) and the recent volumes of Glendenning (2000) and Haensel et al. (2007), as well as other works mentioned in specific places of the text. For an inspiring popular history of compact star research, written by one of its main participants, see Thorne (1994).

2. DEGENERATE FERMIONS

The lowest energy state of a system of fermions is the "Fermi sea" or momentum-space "Fermi sphere", in which N fermions occupy the least energetic orbitals available to them. If the fermions are confined to a real-space volume V and otherwise non-

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interacting, they will agglomerate in momentum space in a sphere around $\vec{p} = 0$ with radius p_F (the "Fermi momentum"), so the volume occupied in 6-dimensional phase space (x, y, z, p_x, p_y, p_z) is $\mathcal{V}_6 = V \times (4\pi p_F^3/3)$. It can be shown (e. g., by solving the Schrödinger or Dirac equation in a box) that there are h^{-3} single-particle orbitals per unit phase-space volume, where $h = 2\pi\hbar$ is Planck's constant. For spin-1/2 fermions such as electrons, protons, or neutrons, Pauli's exclusion principle allows two particles to be put in each orbital (with spin projections $s_z = \pm 1/2$ on an arbitrary axis), thus the total occupied phase-space volume must also be equal to $\mathcal{V}_6 = Nh^3/2$. Equating both expressions for $\mathcal{V}_6$, one finds the Fermi momentum as a function of the number density of fermions, $n \equiv N/V$,

$$p_F = \hbar(3\pi^2 n)^{\frac{1}{3}}\,, \tag{1}$$

which holds regardless of how relativistic the particles are.

Global properties of the fermion system, such as its total energy E, can be calculated as sums over all particles, i. e., integrals over momentum space,

$$E = \frac{2V}{h^3}\int_{|\vec{p}|<p_F} d^3p\ \varepsilon(\vec{p})\,, \tag{2}$$

where one should generally use the relativistic energy-momentum relation $\varepsilon(p) = [(mc^2)^2 + (cp)^2]^{1/2}$, where m is the mass of the fermions and c is the speed of light. These integrals can be done analytically (see Shapiro & Teukolsky 1983); however, the calculation greatly simplifies in both the non-relativistic limit ($p \ll mc$), in which $\varepsilon(p) \approx mc^2 + p^2/(2m)$, and in the extreme relativistic limit ($p \gg mc$), in which $\varepsilon(p) \approx cp$. In these limits, the pressure $P = -(\partial E/\partial V)_N$ takes a polytropic form $P \propto n^\gamma$, respectively

$$P \approx (3\pi^2)^{\frac{2}{3}}\frac{\hbar^2}{2m}\ n^{\frac{5}{3}}\,, \tag{3}$$

and

$$P \approx (3\pi^2)^{\frac{1}{3}}\frac{\hbar c}{4}\ n^{\frac{4}{3}}\,. \tag{4}$$

(For rough estimates, it is useful to remember that, aside from a factor $0.4 - 0.25$, the pressure is just n times the kinetic part of the Fermi energy.)

3. WHITE DWARFS AND NEUTRON STARS

Since the temperature is essentially zero (compared to the Fermi energies), the structure of degenerate stars is determined by only two of the standard four equations of stellar structure, whose Newtonian (weak-gravity) form is

$$\frac{dP}{dr} = -\frac{GM(r)\rho}{r^2}\,, \qquad \frac{dM(r)}{dr} = 4\pi r^2\rho\,, \tag{5}$$

which can be combined to yield an order-of-magnitude expression for the pressure required to sustain a star of given total mass M and characteristic mass density ρ against its own gravity,

$$P \sim GM^{2/3}\rho^{4/3}. \tag{6}$$

The ratio ρ/n is generally constant, taking similar values:

- $\rho/n \approx Am_u/Z \approx 2m_u$ in white dwarfs, in which most of the mass density is provided by ions of mass Am_u (where m_u is the atomic mass unit) and charge Ze, while the degeneracy pressure is due to the electrons, and
- $\rho/n \approx m_n \approx m_u$ in neutron stars, where neutrons (of mass m_n) dominate both the pressure and the mass density.

In the low-density, non-relativistic limit of equation (3), the latter can be combined with equation (6) to estimate an equilibrium radius for the star,

$$R \sim \left(\frac{n}{\rho}\right)^{\frac{5}{3}}\frac{\hbar^2}{Gm}M^{-\frac{1}{3}}\,. \tag{7}$$

Due to the widely different fermion masses m ($m_n \approx 1839\, m_e$), the radii for white dwarfs and neutron stars are very different ($\sim 10^4$ km for the former and ~ 10 km for the latter, at a similar, solar mass $M \sim M_\odot$). Equation (7) shows that, as the mass increases, the stars become smaller and denser and thus their matter more relativistic. In the limit of high density, the hydrostatic pressure of equation (6) needs to be provided by extremely relativistic particles (equation [4]), yielding a maximum mass

$$M_{\max} \sim \left(\frac{\hbar c}{G}\right)^{\frac{3}{2}}\left(\frac{n}{\rho}\right)^2 \sim \left(\frac{m_{\rm Pl}}{m_u}\right)^3 m_u \sim M_\odot\,, \tag{8}$$

where $m_{\rm Pl} = (\hbar c/G)^{\frac{1}{2}}$ is the "Planck mass", a natural mass scale for quantum gravity. For white dwarfs, this maximum mass is the well-known "Chandrasekhar limit" $M_{\rm Chandra} \approx 1.4\ M_\odot$, beyond which a degenerate stellar core collapses to form a neutron star or black hole. Note that the formation of a neutron star involves the release of the binding energy $\sim GM^2/R \sim 0.2\ Mc^2$ as neutrinos (and perhaps gravitational waves), so a neutron star mass can be lower than the Chandrasekhar limit by a corresponding amount. In fact, a neutron star mass as low as 1.25 $M_\odot$ was measured in the recently discovered double pulsar system (Lyne et al. 2004).

White dwarfs are well described by Newtonian gravity, and the main inter-particle forces are electrostatic, for which the interaction energies are much smaller than the kinetic energies, so the physics described above suffices to obtain an accurate description of the structure of a white dwarf. In neutron stars, such estimates are generally not accurate, since at super-nuclear densities the baryons (neutrons and protons) interact strongly with each other (interaction energies comparable to the kinetic energies), and the effects of General Relativity become important (see below). Instructions for students to construct more realistic neutron star models numerically have been given by Silbar & Reddy (2004) and Sagert et al. (2006). Equation (8) suggests a maximum mass for neutron stars at most a few times larger than $M_{\rm Chandra}$. This is likely to be correct, but its precise value is not known, mainly due to the uncertain strong interactions among the neutrons.

Given the basic structural parameters of the stars (here taken as $M = M_\odot$ and $R = 10$ km), one can obtain several other interesting quantities. The escape speed $v_{\rm esc} = (2GM/R)^{\frac{1}{2}} = 0.5\ c$ confirms that neutron star gravity is "strong" and general-relativistic effects can be important. Their minimum allowed rotation period (at which the centrifugal force on the equator equals the gravitational force) $P_{\rm min} = 2\pi[R^3/(GM)]^{\frac{1}{2}} = 0.5$ ms, is much shorter than for any other kind of stars (including white dwarfs, for which $P_{\rm min} \sim 20$ s), a crucial argument in identifying pulsars as neutron stars.

Finally, one may estimate a safe upper bound on a typical magnetic field B in a neutron star by requiring that the magnetic energy be lower than the gravitational binding energy (so the Lorentz force does not exceed the gravitational force)[2],

$$\frac{B^2}{8\pi}\frac{4\pi R^3}{3} < \frac{GM^2}{R}\,, \tag{9}$$

yielding

$$B_{\rm max} \sim 10^{18}\ {\rm G}\,. \tag{10}$$

4. WHY *NEUTRON* STARS?

The name "neutron stars" incorrectly suggests stars composed exclusively of neutrons. However, additional particles inside these stars play a crucial role. A neutron (n) in vacuum decays by the weak interaction process $n \to p + e + \bar\nu_e$ (beta decay) into a proton (p), an electron (e), and an electron antineutrino ($\bar\nu_e$), with a half-life close to 15 minutes. This is impeded in very dense matter by the Pauli exclusion principle: If all the low-energy proton and electron states are already occupied, only suffiently energetic neutrons can decay. On the other hand, if many protons and electrons are present, some of these will be energetic enough to combine into neutrons by inverse beta decay, $p+e \to n+\nu_e$, where ν_e stands for an electron neutrino. In a neutron star, the neutrons and protons will be confined by gravity, the electrons by the electrostatic potential of the protons (e. g., Reisenegger et al. 2006), while neutrinos and antineutrinos are unbound and escape, contributing to the cooling of the star (e. g., Yakovlev et al. 2001). Direct and inverse beta decays are in balance if the chemical potentials[3] of neutrons (μ_n), protons (μ_p), and electrons (μ_e) satisfy the relation $\mu_n = \mu_p + \mu_e$, which forces the coexistence of a small fraction (few percent, but density-dependent) of charged particles with a much larger number of neutrons (e. g., Shapiro & Teukolsky 1983). Additional particles (both charged and uncharged) can appear by other weak decay processes at densities higher than typical nuclear densities.

In addition to stabilizing the neutrons, the charged particles play two important roles regarding the magnetic fields and their evolution. Being charged, these particles can generate electrical currents, which support potentially very strong magnetic fields. In addition, since the proton fraction $Y \equiv n_p/n$ depends on density (n_i stands for the number density of particle species $i = n, p, e$, and $n \equiv n_n + n_p$ is the total baryon density), neutron star matter is inhomogeneous, stabilizing it with respect to convective overturn (Pethick 1992; Reisenegger & Goldreich 1992; Reisenegger 2001a). As discussed below, this is likely to have an important stabilizing effect on magnetic field configurations.

5. FARADAY'S LAW AND ASTROPHYSICAL MAGNETIC FIELDS

Long-lived magnetic fields are ubiquitous in the Universe, and neutron stars are no exception. These nearly static magnetic fields must have currents acting as sources, according to Ampére's law,

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}\,. \tag{11}$$

Currents imply charges of one sign (i. e., electrons) moving with respect to those of the opposite sign

[2] As customary in theoretical astrophysics, I use Gaussian cgs units, in which the unit of magnetic field is the gauss, $\mathrm{G} \equiv \mathrm{erg}^{\frac{1}{2}}\mathrm{cm}^{-\frac{3}{2}}$.

[3] At zero temperature, these chemical potentials reduce to the respective Fermi energies.

(protons or other ions). On time scales much shorter than the observed lifetimes of the fields, these particles suffer Coulomb collisions, which would damp their relative motion, if it were purely due to inertia.

Thus, in order for the magnetic fields to survive, a much larger "inertia" is required, which is provided by Faraday's induction law: Any change of a magnetic field induces an electric field,

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla \times \vec{E}\,. \tag{12}$$

The electric field keeps the current going in spite of the frequent collisions, as described by Ohm's law,

$$\frac{\vec{j}}{\sigma} = \vec{E}\,, \tag{13}$$

where the conductivity σ is inversely proportional to the collision rate. Combining equations (11), (12), and (13), one obtains a diffusion equation for the magnetic field,

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \left(\frac{c^2}{4\pi\sigma}\nabla \times \vec{B}\right)\,, \tag{14}$$

suggesting an Ohmic (or resistive) decay time $t_B \sim 4\pi\sigma R^2/c^2$, where R is a characteristic length scale, such as a stellar radius. In astrophysical plasmas, the length scales are enormously larger than in laboratory conditions, which allows astrophysical magnetic fields to be so long-lived. In the particular case of neutron stars, the Pauli principle also makes it hard for particles to be scattered into different quantum states, therefore enhancing the conductivity. Based on this, Baym et al. (1969) showed that neutron star magnetic fields might live longer than a Hubble time.

The only "loophole" that allows for a significant evolution of the magnetic field is to allow for a velocity field $\vec{v}_e$ of the electrons in the reference frame of interest (i. e., the center-of-mass frame of a star). This velocity adds a magnetic force term $(1/c)\vec{v}_e \times \vec{B}$ on the right-hand side of equation (13), and correspondingly a term $\nabla \times (\vec{v}_e \times \vec{B})$ on the right-hand side of equation (14), which can be interpreted as an advection of the magnetic field lines by the motion of the electrons. In most astrophysical contexts, this motion is shared by all the other particles, corresponding to an ideal magneto-hydrodynamic motion (e. g., Kulsrud 2005). However, in neutron stars, other variants are possible (Goldreich & Reisenegger 1992), such as a motion of all the charged particles with respect to the neutrons ("ambipolar diffusion", e. g., Hoyos et al. 2008), or by only the most mobile charge carrier ("Hall drift"; e. g., Reisenegger et al. 2007), each of which has quite distinctive properties.

6. MAGNETIC FIELDS IN NEUTRON STARS

Many neutron stars are detected as pulsars, whose regular pulsations in the radio, X-ray, and/or optical bands are produced by a strong magnetic field turning around at the stellar rotation period P. These periods slowly increase in time, i. e., the neutron stars lose rotational energy, probably through magnetic coupling with their surroundings. Modelling this coupling as electromagnetic radiation from a dipole rotating in vacuum, oriented orthogonally to the rotation axis, one can infer the surface magnetic field strength $B \propto \sqrt{P\dot{P}}$, where $\dot{P}$ is the time-derivative of the rotation period (e. g., Shapiro & Teukolsky 1983). Inferred fields range from 10^8 G in millisecond pulsars up to 10^{15} G in soft gamma-ray repeaters (SGRs), the latter being the strongest magnetic fields known in the Universe, but still 3 orders of magnitude weaker than the strongest that might conceivably be present in neutron stars according to equation (10). In this (dynamical) sense, neutron star magnetic fields are quite weak, as they are in all other stars known so far. In spite of this, the magnetic field may be the main agent breaking the axial symmetry of the mass distribution in a rotating neutron star, in this way producing precession (Wasserman 2003), as appears to be observed in some pulsars, and gravitational waves, which might quickly reduce the rotation rate of newborn neutron stars (Cutler 2002).

In radio pulsars, the rotational energy loss can account for the whole observed energy output (relativistic particles and electromagnetic radiation). For the strongly magnetized, but slowly rotating SGRs and anomalous X-ray pulsars (AXPs), however, the observed X-ray luminosity is much larger than the rotational energy loss rate, so an additional source of energy is required, the most likely being the decay of their magnetic field (Thompson & Duncan 1996). This would make these objects be the only known magnetically powered stars, or "magnetars". An interesting way of probing the strong magnetic fields inside these objects appears to be the quasi-periodic oscillations recently detected following two large flares of SGRs and interpreted as crustal shearing modes coupled to Alfvén waves travelling through the stellar core (Levin 2007).

In very old neutron stars, such as millisecond pulsars and low-mass X-ray binaries, the magnetic field is $< 10^9$G, weaker than in young neutron stars, such as radio pulsars and high-mass X-ray binaries ($\sim 10^{11-14}$G), suggesting that the magnetic field strength decays with time, perhaps induced by accretion of matter from the binary companion (e. g., Payne & Melatos 2007, and references therein). Magnetic field decay within the population of single radio pulsars has also been suggested by some authors (Ostriker & Gunn 1969; Narayan & Ostriker 1990) but disputed by others (Bhattacharya et al. 1992; Faucher-Giguère & Kaspi 2006), and does not seem to be well established.

7. ORIGIN OF THE MAGNETIC FIELD

A natural hypothesis to explain the origin of the strong fields observed in neutron stars is the compression of the magnetic flux already present in the progenitor stars. It led Woltjer (1964) to predict field strengths of 10^{14-16} G, before any neutron stars had been identified observationally. Many authors (e. g., Ruderman 1972; Reisenegger 2001b; Ferrario & Wickramasinghe 2005a,b, 2006) have pointed out that the distribution of magnetic fluxes is very similar in magnetic A and B stars, white dwarfs, and neutron stars, in this way providing support for the hypothesis of the magnetic fluxes being generated on or even before the main-sequence stage and then inherited by the compact remnants.

On the other hand, Thompson & Duncan (1993) suggested that newborn neutron stars are likely to combine vigorous convection and differential rotation, making a dynamo process likely to operate in them. They predicted fields up to 10^{15-16} G in neutron stars with few-millisecond initial periods, and suggested that such fields could explain much of the phenomenology associated with SGRs and AXPs (Duncan & Thompson 1992; Thompson & Duncan 1995, 1996), some of which were later confirmed to spin down at a rate consistent with a strong dipole field (10^{14-15}G; Kouveliotou et al. 1998; Woods et al. 1999).

Of course, the two processes are not mutually exclusive. A strong field might be present in the collapsing star, but later be deformed and perhaps amplified by some combination of convection, differential rotation, and magnetic instabilities (Tayler 1973; Spruit 2002). The relative importance of these ingredients depends on the initial field strength and rotation rate of the star. For both mechanisms, the field and its supporting currents are not likely to be confined to the solid crust of the star, but distributed in most of the stellar interior, which is mostly a fluid mixture of neutrons, protons, electrons, and other, more exotic particles.

8. PERSISTENT, ORDERED FIELD STRUCTURES

The magnetic fields of neutron stars, like those of upper main sequence stars and white dwarfs, appear to be ordered (with a roughly dipolar external configuration) and persistent (for much longer than a solar cycle, perhaps for the entire existence of these stars). As mentioned above, their magnetic flux distributions are similar, which also implies that their ratios of magnetic to gravitational energy (or magnetic stress to fluid pressure) are similarly small in all of them. The Lorentz force is much smaller than the pressure gradient and the gravitational force that are dominant in establishing the hydrostatic equilibrium in the star. Therefore, the hydro-magnetic equilibrium state can be considered as a small perturbation to an unmagnetized, "background" hydrostatic equilibrium (denoted by a subscript "0"), in which the (conceptual) introduction of the magnetic field forces the fluid to displace from its "initial" position, $\vec{r} \to \vec{r} + \vec{\xi}(\vec{r})$. This causes small perturbations of density, which are customarily described in two complementary ways:

- Eulerian perturbations, which compare the density at the same point in space, before and after the perturbation,

$$\delta\rho(\vec{r}) = \rho(\vec{r}) - \rho_0(\vec{r}) \,, \tag{15}$$

- Lagrangian perturbations, the change in the same fluid element before and after being displaced,

$$\Delta\rho(\vec{r}) = \rho(\vec{r} + \vec{\xi}[\vec{r}]) - \rho_0(\vec{r}) \,. \tag{16}$$

These two descriptions are related by

$$\Delta\rho - \delta\rho = \vec{\xi} \cdot \nabla\rho_0 = \frac{d\rho_0}{dr}\xi_r \,, \tag{17}$$

and exactly the analogous relations hold for the Eulerian and Lagrangian perturbations of the pressure, δP and ΔP. These perturbations must satisfy the force balance condition

$$\frac{\vec{j} \times \vec{B}}{c} - \nabla\delta P - \delta\rho\ \nabla\psi_0 = 0 \,, \tag{18}$$

where $\psi_0(r)$ is the background gravitational potential (assumed to be unperturbed, in the so-called "Cowling approximation").

Another important, shared property of these stars is that a large part (if not all) of their interior is stably stratified (i. e., it resists convective overturn). In the case of upper main-sequence envelopes and white dwarfs, this is because of a radially increasing entropy; in the case of neutron stars, because of a radial dependence in the fraction of protons, electrons, and possibly other particles. This has the consequence that the adiabatic sound speed, relating the Lagrangian pressure and density perturbations of a given fluid element (that conserve entropy and composition), $c_s^2 = \Delta P/\Delta\rho$, is larger than the background derivative $dP_0/d\rho_0 = (dP_0/dr)/(d\rho_0/dr)$, in which entropy or composition are changing (e. g., Reisenegger & Goldreich 1992). The Lagrangian perturbations can be directly related to the divergence of the fluid displacement field $\vec{\xi}(\vec{r})$ causing them,

$$\frac{\Delta P}{c_s^2} = \Delta\rho = -\rho_0 \nabla \cdot \vec{\xi}. \tag{19}$$

From all of the above, one obtains that the Eulerian perturbations $\delta\rho$ and δP are linearly independent combinations of $\nabla \cdot \vec{\xi}$ and ξ_r,

$$\delta\rho = -\rho_0 \nabla \cdot \xi - \frac{d\rho_0}{dr}\xi_r\,, \tag{20}$$

$$\delta P = -\rho_0 c_s^2 \nabla \cdot \xi - \frac{dP_0}{dr}\xi_r\,, \tag{21}$$

and can therefore be regarded as independent variables.

Thus, a given, sufficiently weak magnetic field $\vec{B}(\vec{r})$ corresponds to an equilibrium configuration if two independent scalar functions δP and $\delta\rho$ can be found that satisfy equation (18). Since the latter is a 3-component vector equation, this will not generally be possible. Therefore, it imposes a condition on $\vec{B}$ that can be written as

$$\hat{r} \cdot \nabla \times [(\nabla \times \vec{B}) \times \vec{B}] = 0\,, \tag{22}$$

a single, scalar condition on the magnetic field. This condition is much less restrictive than the often imposed force-free condition, $(\nabla \times \vec{B}) \times \vec{B} = 0$ (Broderick & Narayan 2008), relevant for the opposite limit of dynamically dominant fields, or even the condition $\nabla \times [(\nabla \times \vec{B}) \times \vec{B}/\rho] = 0$ (Haskell et al. 2008), required only for barotropic fluids with a unique pressure-density relation $\rho(P)$, which are not stably stratified.

The assumption of axial symmetry, with

$$\vec{B} = \nabla \times [\alpha(r,\theta)\nabla\phi] + \beta(r,\theta)\nabla\phi\,, \tag{23}$$

where $\nabla\phi = \hat{\phi}/(r\sin\theta)$, considerably simplifies the problem of constructing equilibria, since in this case equation (18) implies that the Lorentz force can have no azimuthal component, $(\vec{j} \times \vec{B})_\phi = 0$, therefore surfaces of constant β coincide with those of constant α (i. e., $\beta = \beta[\alpha(r,\theta)]$). These "allowed" magnetic field configurations produce two independent force components in the meridional plane, which can generally be cancelled by an appropriate choice of the functions δP and $\delta\rho$, so no additional conditions are imposed on $\vec{B}(\vec{r})$ to correspond to an equilibrium.

Of course, in order to be viable, a certain magnetic field configuration must correspond to a *stable* equilibrium, which is much more difficult to characterize and still largely an open problem. Analytic attempts to search for stable magnetic field configurations have failed, only yielding the general result that both purely toroidal fields $\vec{B} = \beta(r,\theta)\nabla\phi$ and purely poloidal fields $\vec{B} = \nabla \times [\alpha(r,\theta)\nabla\phi]$ are unstable (Tayler 1973; Flowers & Ruderman 1977), and the speculation that linked toroidal and poloidal fields might stabilize each other, yielding a stable equilibrium (Prendergast 1956; Wright 1973). Recent MHD simulations (Braithwaite & Spruit 2004, 2006; Braithwaite & Nordlund 2006) have shown initially complex, "random" magnetic fields to evolve on an Alfvén-like timescale into a roughly axisymmetric, linked poloidal-toroidal configuration that persisted for a resistive timescale and thus might be a good approximation to the field structures in upper-main sequence, white dwarf, and neutron stars.

Once a stable, ideal-MHD equilibrium magnetic field has been established, it will survive for many Alfvén times, but not forever, since there are several dissipative processes by which it could evolve on long time scales (Goldreich & Reisenegger 1992; Reisenegger 2007; Reisenegger et al. 2007; Hoyos et al. 2008), possibly matching the times on which magnetar fields appear to decay (Thompson & Duncan 1996).

9. CONCLUSIONS

Neutron stars are fascinating objects with extreme properties, which include the strongest magnetic fields in the Universe. Nevertheless, these share properties with those of other stars, among these, that they are weak in the sense of producing only small disturbances to the structure of the respective stars. Some progress has been made in understanding possible magnetic field configurations and their evolution, but there is still much left to do.

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