where

$$\hat{Q}_{xy} = -\frac{1}{8\alpha(1+\alpha)^2}\frac{dq}{dA}\exp(-y). \tag{12}$$

In these units,

$$q(A) = \frac{1}{2}\frac{d\tilde{m}}{dA}\left[\int_0^{\lambda_x/2} dx \frac{\partial y(x,A)}{\partial A}\exp[-y(x,A)]\right]^{-1}, \tag{13}$$

where

$$\frac{d\tilde{m}}{dA} = -2\lambda_x(1+\alpha)A. \tag{14}$$

Under a constant gravity condition, the vector potential in the plane-parallel unperturbed state is

$$A_0(y) = \exp\left(-\frac{y}{2(1+\alpha)}\right), \tag{15}$$

and the source term of equation (11) is

$$\hat{Q}^{(i)}_{xy} = \frac{1}{4(1+\alpha)^2}\exp\left(-\frac{y}{2(1+\alpha)}\right). \tag{16}$$

In this initial state, the relation between A and q is $A_0(q) = q^{-1/(2\alpha)}$ (see also M74).

3.2. *The axisymmetric model*

Now, we consider a three-dimensional geometry with axial symmetry about the z-axis and longitudinal magnetic field ($B_\phi = 0$). The magnetic field is related to the potential vector $\vec{A} = A(z,R)\hat{e}_\phi$ by $B_z = R^{-1}\partial(RA)/\partial R$ and $B_R = -\partial A/\partial z$. Under axisymmetric disturbances, the magnetic flux $\Phi(R,z) \equiv RA$ is constant on a magnetic surface and is also a constant of the motion. In this geometry, the function q defined in equation (7) is given by

$$q(\Phi) = \frac{c_s^2}{2\pi}\frac{dm}{d\Phi}\left[\int_0^{\frac{\lambda_z}{2}} dz \frac{\partial R^2(z,\Phi)}{\partial\Phi}\exp\left(-\frac{\psi(z,\Phi)}{c_s^2}\right)\right]^{-1}, \tag{17}$$

where λ_z is the longitudinal perturbation wavelength and dm is the mass in a flux tube between Φ and $\Phi + d\Phi$ and between $z = 0$ to $z = \lambda_z$. Again, q is a constant along field lines in equilibrium configurations. Under flux-freezing conditions, the mass-to-flux ratio is a constant of the motion, which can be determined from the initial configuration by:

$$\frac{dm}{d\Phi} = 2\pi\lambda_z\frac{\rho_0}{B_0}. \tag{18}$$

The balance of forces (equation 5) in the final magnetic configuration can be rewritten in terms of Φ as

$$\nabla^2\Phi - \frac{2}{R}\frac{\partial\Phi}{\partial R} = -4\pi R^2\frac{dq}{d\Phi}\exp\left(-\frac{\psi}{c_s^2}\right), \tag{19}$$

(Mouschovias 1976; Nakano 1979). Equation (19) together with equation (17) enable us to derive $\Phi(z,R)$ once the boundary conditions are specified. The magnetic field can be found just by simple derivation: $B_z = R^{-1}\partial\Phi/\partial R$ and $B_R = -R^{-1}\partial\Phi/\partial z$.

The computational domain is a rectangular box of $0 < z < Z_{\max}$ and $0 < R < R_{\max}$. Because of the symmetries of the initial and final states, the length of the computation box in the z-direction, $Z_{\max}$, is taken to be $\lambda_z/2$, half the wavelength of the perturbation. Periodicity in z translates into

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0,Z_{\max}} = 0. \tag{20}$$

The definition of Φ means that $\Phi(0,z) = 0$ at any time. Assuming that the deformation of field lines can be neglected on the surface of the cylinder, we impose the condition that $\Phi(z,R_{\max}) = \Phi_0(R_{\max})$.

The equivalent condition of constant gravity is to assume that the radial gravitational field is constant, i.e. $\vec{g} = -g\hat{e}_R$, with g a positive constant acceleration. We may use the units for distance, velocity and density described in the previous subsection. The magnetic flux will be measured in units of $-4H^2B_0(0)$. Equation (19) in dimensionless form is:

$$\nabla^2\Phi - \frac{2}{R}\frac{\partial\Phi}{\partial R} = Q_{zR}(z,R;\alpha), \tag{21}$$

with

$$Q_{zR} = -\frac{R^2}{32\alpha(1+\alpha)^4}\frac{dq}{d\Phi}\exp(-R), \tag{22}$$

and

$$q(\Phi) = \frac{1}{2\pi}\frac{dm}{d\Phi}\left[\int_0^{\lambda_z/2} dz \frac{\partial R^2(z,\Phi)}{\partial\Phi}\exp[-R(z,\Phi)]\right]^{-1}, \tag{23}$$

where the dimensionless mass-to-flux is

$$\frac{dm}{d\Phi} = -8\pi(1+\alpha)^2\lambda_z\frac{\rho_0}{B_0} \tag{24}$$

$$= -8\pi(1+\alpha)^2\lambda_z\exp\left(-\frac{R(\Phi)}{2(1+\alpha)}\right). \tag{25}$$

By comparing equations (21)–(22) with equations (11)–(12), we see that the dynamical variable Φ also satisfies a nonlinear elliptical differential equation, but with a differential operator and source term different from those of the equation governing A in the Cartesian model.

In order to gain some physical insight, it is useful to compute the function $q(\Phi)$ in the initial equilibrium configurations which, of course, are solutions